May 2015

Self-Force on Accelerated Particles

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by

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A Dissertation Submitted in
Partial Fulfillment of the
Requirements for the degree of

Doctor of Philosophy
in Physics

at
The University of Wisconsin–Milwaukee
May 2015

ABSTRACT
SELF FORCE ON ACCELERATED PARTICLES
The University of Wisconsin–Milwaukee, 2015
Under the Supervision of Professor Alan Wiseman

The likelihood that gravitational waves from stellar-size black holes spiraling into a supermassive black hole would be detectable by a space based gravitational wave observatory has spurred the interest in studying the extreme mass-ratio inspiral (EMRI) problem and black hole perturbation theory (BHP). In this approach, the smaller black hole is treated as a point particle and its trajectory deviates from a geodesic due to the interaction with its own field. This interaction is known as the gravitational self-force, and it includes both a damping force, commonly known as radiation reaction, as well as a conservative force. The computation of this force is complicated by the fact that the formal expression for the force due to a point particle diverges, requiring a careful regularization to find the finite self-force.

This dissertation focuses on the computation of the scalar, electromagnetic and gravitational self-force on accelerated particles. We begin with a discussion of the "MiSaTaQuWa" prescription for self-force renormalization [19, 20] along with the refinements made by Detweiler and Whiting [36], and demonstrate how this prescription is equivalent to performing an angle average and renormalizing the mass of the particle. With this background, we shift to a discussion of the "mode-sum renormalization" technique developed by Barack and Ori [1], who demonstrated that for particles moving along a geodesic in Schwarzschild spacetime (and later in Kerr spacetime), the regularization parameters can be described using only the leading and subleading terms (known as the A and B terms). We extend this to demonstrate that this is true for fields of spins 0, 1, and 2, for accelerated trajectories in arbitrary spacetimes.

Using these results, we discuss the renormalization of a charged point mass moving through an electrovac spacetime; extending previous studies to situations in which the gravitational and electromagnetic contributions are comparable. We renormalize by using the angle average plus mass renormalization in order to find the contribution from the coupling of the fields and encounter a striking result: Due to a remarkable cancellation,

the coupling of the fields does not contribute to the renormalization. This means that the renormalized mass is obtained by subtracting (1) the purely electromagnetic contribution from a point charge moving along an accelerated trajectory and (2) the purely gravitational contribution of an electrically neutral point mass moving along the same trajectory. In terms of the mode-sum regularization, the same cancellation implies that the regularization parameters are merely the sums of their purely electromagnetic and gravitational values.

Finally, we consider the scalar self-force on a point charge orbiting a Schwarzschild black-hole following a non-Keplerian circular orbit. We utilize the techniques of Mano, Suzuki, and Takasugi [2] for generating analytic solutions. With this tool, it is possible to generate a solution for the field as a series in the Fourier frequency, which allows researchers to naturally express the solutions in a post Newtonian series (see Shah et. al. [3]). We make use of a powerful insight by Hikida et. al. [4, 5], which allows us to perform the renormalization analytically. We investigate the details of this procedure and illuminate the mechanisms through which it works. We finish by demonstrating the power of this technique, showing how it is possible to obtain the post Newtonian expressions by only explicitly computing a handful of ℓ modes.

To my wonderful wife, Whitney

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Acknowledgments

I first want to thank David Abineri, my high school calculus and physics teacher who introduced me to the beauty of physics. The problem solving abilities that I developed in his classes have been invaluable. Similarly, I want to thank Dr. Jan Kmetko who helped me learn how to solve problems physically instead of through brute force mathematics.

I would like to thank my advisors, Alan Wiseman and John Friedman. Alan's refusal to accept explanations that were only "pretty good" forced me to dig deeper in order to find an explanation that would meet his strict requirements. Working with John helped me develop the organization of thought necessary to successfully verify long calculations. I would also like to thank John in particular for the incredible support he has given me and for his tireless work on my behalf.

Several fellow graduate students have been invaluable allies throughout graduate school. My classmates Sydney Chamberlain, Les Wade and Maddie Wade were incredible friends and our 'homework parties' in our first and second years helped me solidify my understanding of the fundamentals and saved me from hours of fruitless labor. My friendship with Alex Urban has helped me get through some of my hardest moments as a graduate student. Eric Van Oeveron has been an incredible help in recent months as he has cleaned up our results and made a few timely insights which has helped to clarify some subtle issues.

Most importantly, I'd like to thank my family for their support through everything. I would like to thank my mother and father-in-law for both their belief in me and their useful insights into the world of academia. My parents have always given their all to give me the best opportunities and the self-confidence to take them. As I complete my formal education I look back on everything and truly appreciate the sacrifices they have willingly made on my behalf. My sister has always been helpful in making me think about the big picture, both in my work and in life in general. Most of all, I would like to thank my wife, Whitney, whose love and devotion have kept me going through the hardest moments in life. Her patience and understanding (and editorial skills) have been invaluable in the recent months as I have waded through the gauntlet of job applications, conference talks and writing. Finally, I would like to acknowledge my cats, Guenwhyvar,

who once corrected my work, and Mirax, who has been both my constant companion and the biggest hurdle during my research.

Chapter 1

Introduction: Binary Systems and Self-force

1.1 A Brief Overview of the General Relativistic Two Body Problem

The study of binary systems in general relativity is a problem of great interest, as it is one of the simplest astrophysically relevant systems that can produce gravitational waves. Because of the non-linearity of Einstein's equations, this problem is not trivially solvable. The efforts to study these systems have spawned numerous approximation techniques and numerical tools, each with its own strengths and weaknesses.

This dissertation will focus on black hole perturbation theory (BHP) one of the four main approaches to studying the general relativistic two body problem. Before discussing this approach it is useful to consider the other three approaches commonly used in order to understand how results from BHP fit into the study of binary systems.

The application of the post-Newtonian (pN) approximation to binary systems is widely used to model many systems of astrophysical interest. In the pN approximation, one expands the metric and particle trajectory as a series in a small parameter epsilon for which the ratio v/c of the speed of the particle to the speed of light is of order epsilon. Therefore, at zeroth order in the series, the system is described by Newtonian physics, and the higher orders are corrections due to special and general relativity. This approximation applies to systems of all mass ratios, but it breaks down in the high-speed and strong-field regimes[1].

Another approach is numerical relativity (NR) which involves using computers to solve the full, non-linear Einstein equations, and then, using this information, evolve the system. In one very real sense this approach "truly solves" the system instead of considering a perturbative series solution, and is thus preferable to the perturbative approaches. This powerful technique is limited only by the power of our computers and as such will become progressive stronger as our computational power increases. This approach is computationally very expensive and is not practical for slowly evolving systems (for example, when the two bodies are far away and moving slowly), or when the mass ratios of the two bodies is large (many researchers focus on the regime with ratios between 1:1 and 1:10 [6]).[2]

The third approach is the effective one body approximation (EOB), which maps the dynamics of the two body problem onto an analogous one body problem [8]. This approach draws on information from post Newtonian approximation (pN) , a study of radiation-reaction, and the conservative dynamics of the system. To quote Damour, " one needs to make use of several tools: (i) resummation methods, (ii) exploitation of the flexibility of analytical approaches, (iii) extraction of the non-perturbative information contained in various numerical simulations, (iv) qualitative understanding of the basic physical features which determine the waveform." [9]. This approach has made some outstanding advances in our understanding of binary systems [8] and is still of great interest today. One cannot, however, utilize this approach on its own, as it requires information from both pN approximations as well as from NR (and as we will see BHP can also aid EOB).

[1]That is to say that in these regimes the "small" parameters are not very small, and it is necessary to use more and more corrections in the highly-relativistic regime in order to recover the same accuracy achieved in the non-relativistic regimes

[2] In 2011, Lousto and Zlochower [7] evolved two orbits with the "extreme" mass ratio of 100:1. The difference in the language used between numerical relativists and that of the self-force community (where 'extreme' is typically used to describe mass ratios of 10^6 and higher) is indicative of the preferred regimes of operation for these two techniques.

As mentioned above, this dissertation will focus primarily on black hole perturbation theory. When applied to binary systems, it is assumed that the smaller of the two objects can be treated as a point particle, whose gravitational field is treated as a perturbation to the spacetime curvature generated by the larger body. This technique is therefore strongest when the ratio of the masses is very large, and thus the astrophysical systems best studied with this technique are extreme mass-ratio inspirals or EMRIs. These systems typically consist of a super massive black hole (whose mass we will refer to as M throughout this work) like those predicted to exist at galactic centers and a solar mass black hole (of mass $\mathtt{m}$), giving the mass ratio $\mu = \mathtt{m}/M \propto 10^{-6}$.

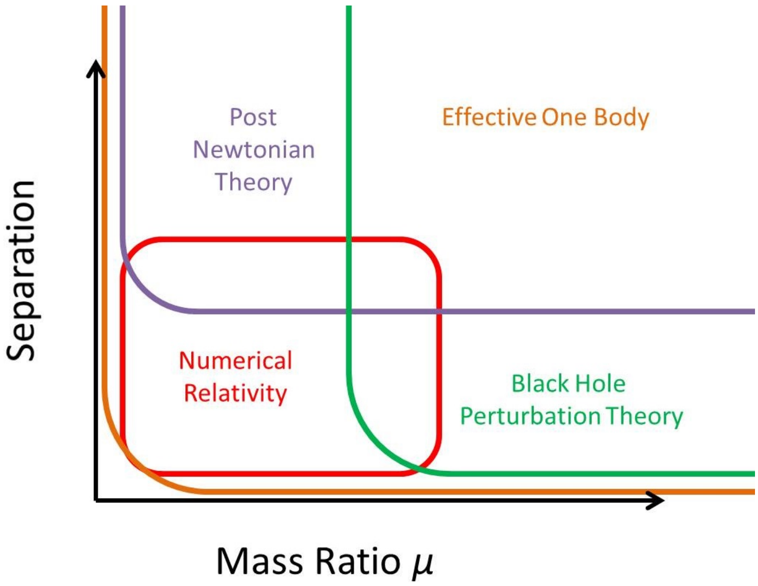

Figure 1: A schematic diagram of the relative ranges of applicability of the four theories used to study binary systems in general relativity. I depict significant overlap between NR, pN, and BHP, the three independent approximations. Significant portions of this entire phase space should, in principle be covered by EOB, which requires input from the other three.

In the gravitational wave community, these systems are of particular interest as objects of study for a laser interferometer space antenna (LISA) or LISA-like device. While BHP is ideally suited to study EMRIs, recent results have shown that BHP has uses in other regimes as well. In 2010, LeTiec et al. [10] demonstrated that BHP can be used to advance our knowledge of post-Newtonian theory, a fact further demonstrated by Shah, et al. [3] who used BHP to find parameters previously overlooked (and since confirmed by Bini and Damour [11] and Blanchet et al. [12]). In another effort, LeTiec et al. [13] used BHP to develop a set of laws of thermodynamics for binary black hole systems, laws which should be applicable for systems of any mass ratio. This was later shown by LeTiec et al. [14], where it is shown that the predictions of BHP for the gravitational binding energy match numerical simulations to a high degree of accuracy for *equal mass binaries*. Furthermore, by using the symmetric mass ratio instead of the canonical mass ratio, results from BHP could be used to help study intermediate mass ratio inspirals (IMRIs).

Two relatively recent results showed how BHP can both inform and be compared to EOB. In one comparison by Sarp et al. [15], BHP was able to provide an analytic fit for the EOB parameter $a(u)$, by making a comparison with Detweiler's gauge invariant quantity, h_{uu}. In another comparison, [16], Bini et al. used EOB to find the same tidal effects predicted by Dolan [17] using BHP. Therefore, BHP is already showing its use both as an informant and as a source for comparison, even for systems where one would naively expect BHP to be unreliable.

Having discussed the role of BHP in the overall study of binary systems, we now focus on the details of this method.

1.2 Black Hole Perturbation Theory and Self-Force

As we discussed before, when we apply BHP to the study of binary systems we treat the smaller black hole as a point particle traveling along a trajectory in the curved spacetime of the larger black hole, and we solve the system perturbatively in a series in the mass ratio, $\mu = \mathtt{m}/M$. To zeroth order in this approximation, the particle travels along a geodesic of the unperturbed background spacetime. The corrections to the particle's trajectory are due to the particle interacting with its own field (of order μ), and we call this interaction the self-force. As our ultimate goal is to obtain the actual trajectory of the black-hole, we wish to develop an expression for this self-force.

Unfortunately, when we try to compute this force, we quickly run into problems. To demonstrate these, let us consider the toy problem of evaluating the scalar self-force on a small, compact body carrying a scalar charge as it orbits a black hole. We will let the body have a scalar charge density ρ, a smooth scalar field ϕ, with an internal stress energy tensor $T_B^{\alpha\beta}$. This body may even be coupled to some other set of fields, described by $T_E^{\alpha\beta}$. Our scalar field satisfies the equation

$$\nabla^\alpha \nabla_\alpha \phi = -4\pi\rho, \tag{1.2.1}$$

and its stress-energy tensor, $T_S^{\alpha\beta}$ is given by

$$T_S^{\alpha\beta} = \frac{1}{4\pi}\left(\nabla^\alpha\phi\nabla^\beta\phi - \frac{1}{2}g^{\alpha\beta}\nabla_\gamma\phi\nabla^\gamma\phi\right). \tag{1.2.2}$$

Therefore, the conservation of stress-energy $\nabla_\beta T^{\alpha\beta} = 0$ tells us that

$$\nabla_\beta\left(T_B^{\alpha\beta} + T_E^{\alpha\beta} + T_S^{\alpha\beta}\right) = 0. \tag{1.2.3}$$

To find the force density exerted on the body by the field, we use this conservation equation and write

$$\nabla_\beta T_B^{\alpha\beta} + \nabla_\beta T_E^{\alpha\beta} = -\nabla_\beta T_s^{\alpha\beta} = \rho\nabla^\alpha\phi. \tag{1.2.4}$$

If we consider the point particle approximation, then we find

$$f_S^\alpha = q\nabla^\alpha\phi. \tag{1.2.5}$$

In the point charge limit, both the field ϕ and its derivative diverge on the world line. [3]

In 1999 Mino, Sasaki, and Tanaka [19] and Quinn and Wald, [20] developed the foundations for regularizing and renormalizing the electromagnetic and gravitational self-force which is today referred to as *MiSaTaQuWa* renormalization. The following year, Quinn [18] adapted the scheme for the scalar self-force. We will discuss this axiomatic procedure in great detail in Chapter 2, but the procedure can be described qualitatively in a very intuitive way.

First of all, we assume that a particle in flat spacetime that interacts with its own half-advanced, half-retarded field will feel no force. Second, since spacetime is locally flat, our point source's field will look like the field from flat spacetime locally, and it is

[3] The above description is paraphrasing Quinn's argument spanning Eqs. (1-5) of [18].

this field which causes the problems in the naive calculation. So, we rid ourselves of the flat spacetime half-advanced − half-retarded force, and then perform an angle average to eliminate any terms that provide a direction dependent force as we approach the particle. The resulting force is well defined at the particle.

This *MiSaTaQuWa* technique provides a good basis for understanding the renormalization of the fields, but it is hampered by the angle-average. While this angle-average provides us with an elegant tool for understanding the method, it is difficult to apply in practice. This has led to greater refinements of the technique which will be discussed in greater detail in Chapters 2 and 3.

1.3 Using Toy Systems

Since the overarching goal of this field of study is to generate the gravitational wave signal from a binary inspiral with a self-force-corrected trajectory, it might seem odd to study the effects of acceleration on the self-force. Since the problem of astrophysical significance concerns the inspiral of a binary black hole system, where both bodies move along geodesics (at zeroth order in the mass ratio), why study the self-force acting on accelerated charges, and why study the scalar and electromagnetic self-forces at all?

Let us answer the second question first. We work with scalar and electromagnetic charges as toy models to help us understand the most daunting aspect of self-force work, namely the renormalization. These toy models both require renormalization very similar to that used for the gravitational self-force, without the additional problems of gauge dependencies and metric reconstruction procedures which arise in the gravitational problem. Indeed, as we will see explicitly in Chapter 2, many of the equations governing the description of the singularity look almost identical, with the primary exception being the number of indices required in describing the fields.

Considering accelerated trajectories has many advantages. One reason to consider accelerated trajectories is that it allows further testing of the renormalization procedures. Therefore, by considering accelerated motion, it is possible to refine previous knowledge of the behavior of the fields due to point sources [21–23], by demonstrating what changes must be made for accelerated motion, and what properties stay the same [24].

Furthermore, studying accelerated motion may allow us to investigate fundamental questions that cannot be approached without it. It has recently been suggested that the self-force might act as a cosmic censor, preventing the overcharging or overspinning of near extreme black holes [25–28]. To test this, one must consider the self-force on a charged, massive particle near an extremal black hole. This charge will move along an accelerated trajectory as the background electromagnetic field acts on it.

A very practical reason for studying the self-force on accelerated motion is that it opens up many more useful comparisons that previously were not possible. For example, if there is a charged particle moving along a circular geodesic in Schwarzschild spacetime, any expression for the field the source produces or the force experienced by the source should reduce to that of a particle carrying a similar charge moving along a circular trajectory in flat spacetime under the limit by taking the limit that the mass of the black hole vanishes.

Unfortunately, any expressions obtained with the assumption of geodesic motion cannot generate this result, as Kepler's law links the particle's speed to the mass of the black hole. In this case, the results would reduce to a particle moving in flat spacetime along a straight line at constant velocity, i.e. moving along a geodesic in flat spacetime. By allowing for accelerated motion, it is possible to make many more comparisons. Without the constraint of geodesic motion, it is possible to verify analytic results by comparing the calculated behavior with a much wider variety of simpler scenarios.

1.4 Structure of the Dissertation

This dissertation will draw heavily from the pair of papers by Linz, Friedman, and Wiseman [24, 29], and finish with the unpublished work performed with Eric Van Oeveron and under the supervision of Alan Wiseman.

We begin in Chapter 2, where we discuss the work done in the first half of [24]. This will include a derivation of the *MiSaTaQuWa* renormalization, the Detweiler and Whiting refinements to the renormalization, and Gralla's angle-average with our generalization. We finish the chapter with the expressions required for renormalizing the scalar, electromagnetic and gravitational self-forces, along with the equations of motion for the point particles.

In Chapter 3, we investigate the mode-sum renormalization procedure–the most widely used and practical procedure used in self-force calculations. In doing so, we present the primary result from [24] demonstrating that a very important aspect of this technique known for geodesic motion in black hole spacetimes, in fact generalizes to general motion in generic smooth spacetimes. Using the results from Chapter 2, we provide the so-called 'regularization parameters' for scalar, electromagnetic, and gravitational self-force renormalizations. We finish the chapter with a discussion of some of the important features of this technique and the analogies between the features of the mode-sum and the corresponding features of the *MiSaTaQuWa* formulation.

In Chapter 4, we present the results from the second paper in the series, [29]. In this work, we used the results from [24] (Chapters 2 and 3) to develop the renormalization scheme for a charged point mass moving through an electrovac spacetime[4]. Renormalizing coupled singular fields requires us to non-trivially extend the results for non-coupled fields and develop the renormalization procedures for the gravitational self-force in non-vacuum spacetimes. The primary results we display here were also found independently by Zimmerman and Poisson [30], different techniques. In section 4.2 we use their results for the scalar field to develop the regularization parameters for renormalization in scalarvac.

In Chapter 5 we will delve into the techniques of MST [2] for generating analytic solutions to the Teukolsky equation. We use this to develop the retarded solutions for the scalar field produced by a charged particle orbiting a Schwarzschild black hole along accelerated, circular trajectories. In doing so, we utilize the insights of Hikida et al. [4, 5] to separate the retarded solutions into two convenient parts. We finish this chapter by computing the damping force experienced by the particle.

In Chapter 6, we use the results from Chapter 5 to compute the conservative self-force on the particle. This is where we make the best use of Hikida's insight in splitting the fields, as splitting the fields allows us to renormalize analytically, as we can perform the

[4]By 'electrovac spacetime,' we are referring to a spacetime with a background electromagnetic field, but that is otherwise vacuum. We assume that the background metric $g_{\mu\nu}$ is a solution to the Einstein Equations sourced by the background electromagnetic field.

summation over all ℓ by using a general ℓ expression for one part of the field. We then discuss how, conversely this technique could be used to determine a pN expansion of the higher order regularization parameters studied by Heffernan et al. [31] [5]. We then demonstrate how this technique compares with numerical studies.

[5]It is not clear exactly how useful this will be– as the whole purpose of finding the higher order regularization parameters is to aid the convergence of the renormalized self-force in cases where it is not possible to sum from $\ell = 0$ all the way to ∞. It might be useful for comparisons between analytic studies and high accuracy numerical work

Chapter 2

The Equations of Motion and Renormalization

The primary difficulties in self-force calculations all arise due to the presence of divergent fields which must be renormalized in order to produce a smooth regular field at the particle which has well defined derivatives from which it is possible to compute the force experienced by the particle. In this Chapter, we [1] will discuss several important advances in self-force renormalization, re-deriving many of their equations in a language that is tailored to discussing the Mode Sum Renormalization of the next Chapter.

The trajectory of a small body moving in a curved spacetime deviates from the geodesic motion of a point particle at linear order in the charge or mass due to the particle's interaction with its own field. Derivations of the corrected trajectory use matched asymptotic expansions and a point-particle limit of a family of finite bodies whose charge, mass and radius simultaneously shrink to zero. These derivations demonstrate that one can describe this corrected first-order trajectory by a renormalized self-force. [2]

In order to recover this renormalized self-force, it is necessary 1) to subtract from the retarded field an expression sharing its same singular structure, and 2) take the finite

[1] This Chapter and the following is based on the work Linz, Friedman, Wiseman [24]. Significant sections of the text will differ only slightly from the original paper.

[2] The most recent and rigorous of these are by Gralla, Harte, and Wald [32, 33] (with a formal proof for an electromagnetic charge), by Pound [34], and by Poisson, Pound and Vega [35], who also review the history and give a comprehensive bibliography.

expression resulting from this subtraction and eliminate all direction dependent pieces. These two steps taken together will produce a smooth[3] field at the particle, referred to as the renormalized field, the derivatives of which provide the renormalized self-force. While all of the procedures used to acquire this renormalized self-force are all based on the *MiSaTaQuWa* procedure, there are differences to each.

The first step mentioned above, subtracting a field (which we will call the singular field from now on) with the same singular behavior from the retarded field, is far from trivial as it involves the subtraction of two divergent quantities. In order to subtract these two quantities, it is necessary to regulate each. That is to say, it is necessary to express these fields in some manner that allows us to compute the difference of two finite quantities, and only then take an appropriate limit to reach the result of the difference of the two divergent quantities.

In this Chapter, we will focus on the regularization technique used by many of the works fundamental to understanding renormalization procedures in general [18, 20, 23, 36]. We will begin by defining the fundamental system of for self-force renormalization and derive the local expansion of the scalar field due to a point source in section. Next, we will explore the axiomatic approach of Quinn [18] (and of Quinn and Wald [20] for electromagnetism and gravity), that gives the famous *MiSaTaQuWa* renormalization procedure in section 2.3. Then we will discuss an important refinement of this technique, introduced by Detweiler and Whiting [36]. Following this, we will discuss an alternative interpretation, championed first by Gralla [23], and the modification to this scheme introduced in my first paper [24]. We will introduce equations of motion and renormalization for the electromagnetic and gravitational self-forces. We complete the Chapter with a discussion of the equations of motion for point particles.

[3] As we will discuss in some detail the precise definition of 'smooth' here is a bit nebulous. When we discuss the mode-sum renormalization techniques, we will treat the renormalized self-force as though it were C^∞. The problem arises from the fact that the precise definition of the renormalized field is not unique, as discussed in section 2.4

2.1 Description of the System

Consider a point particle (a scalar charge q, electric charge e, or mass m) traveling on an accelerated trajectory $z(\tau)$ in a smooth spacetime $(M, g_{\alpha\beta})$, where τ is proper time. Let x be a field point that lies on the spacelike $t = 0$ slice and is in a convex normal neighborhood, C of $z(0)$. We define ϵ to be the geodesic distance from the particle's position at an arbitrary time τ to x; that is, ϵ is the length of the unique geodesic from an arbitrary point on the trajectory $z(\tau)$. After performing the various derivative operations to get to an expression for the singular field and singular force, we will choose the arbitrary point to be $z(0)$. In particular, with an eye to our discussion of the mode-sum schemes in the next Chapter, we will consider ϵ to be the length of the unique geodesic from $z(0)$ to x (see Fig. 2).

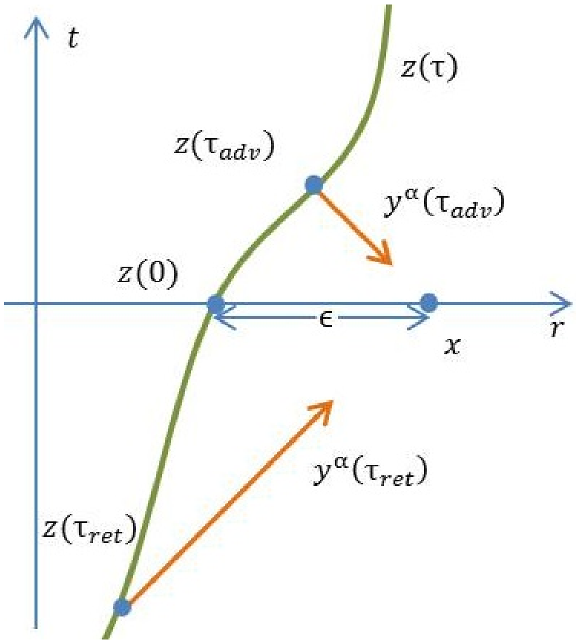

Figure 2: The particle trajectory $z(\tau)$. Two null vectors $y^{\alpha}(\tau_{ret})$ and $y_{\alpha}(\tau_{adv})$ are tangent to future- and past-directed null geodesics from points along the trajectory to a field point x. A geodesic from $z(0)$ to x has length ϵ.

We will restrict the discussion to consider only the scalar self-force. Assume that the scalar field, Φ, obeys the Klein-Gordon equation for a massless field 1.2.1. And the charge density $\rho(x)$ is given by,

$$\rho(x) = q \int d\tau \delta^4(x, z(\tau)). \tag{2.1.1}$$

We will use RNCs about a point $\tau = 0$ of the trajectory and, for mode-sum regularization, spherical coordinates (t, r, θ, ϕ) associated with an arbitrary smooth Cartesian chart. For brevity of notation, assume that $t = 0$ at $\tau = 0$.

2.2 A Local Expansion of the Field

To solve for the singular structure of these fields, it is expedient to use the Hadamard forms of the advanced and retarded Green's functions. Assuming that the events $x, x' \in C$, we write

$$G^{adv/ret}(x, x') = \Theta_{\pm}(x, x') \left[U(x, x')\delta(\sigma(x, x')) - V(x, x')\theta(-\sigma(x, x'))\right], \tag{2.2.1}$$

where $V(x, x')$ and $U(x, x')$ are smooth bi-scalar functions of x and x', and $\sigma(x, x')$ is half the squared length of the geodesic connecting x and x'. The function $\Theta_{\pm}(x, x')$ is unity when x' is in the causal future (past) of the event x for the advanced (retarded) Green's function, and vanishes otherwise.

The retarded solution to Eqs. (1.2.1) and (2.1.1) is given by

$$\begin{aligned} \Phi^{ret} &= q \int d^4x' \sqrt{-g} \int d\tau G^{ret}(x, x') \delta^4(x', z(\tau)), \\ &= q \int d\tau G^{ret}(x, z(\tau)). \end{aligned} \tag{2.2.2}$$

Following Quinn, we split the region of integration into two regions: the part of the trajectory in the normal neighborhood C (where the Hadamard form of the Green's function is valid) and the rest of the trajectory. We choose the event x to be close enough to the trajectory that the events $z(\tau_{adv})$ and $z(\tau_{ret})$ both lie in C, and we denote by $T_{\pm}$ the proper times at which the trajectory intersects the boundary ∂C: The past and future intersection points are respectively $z(T_{-})$ and $z(T_{+})$. The retarded field then takes the form

$$\begin{aligned}
\Phi^{ret} &= q\int_{T_-}^{T_+}\Theta_-(x,z(\tau))\left[U(x,z(\tau))\delta(\sigma(x,z(\tau))) - V(x,z(\tau))\theta(-\sigma(x,z(\tau)))\right]d\tau \\
&+ q\int_{-\infty}^{T_-} G^{ret}d\tau \\
&= q\int_{T_-}^{0}\left[U\delta(\sigma) - V\theta(-\sigma)\right]d\tau + q\int_{-\infty}^{T_-}G^{ret}d\tau, \qquad (2.2.3)
\end{aligned}$$

where we have suppressed the arguments of the biscalar functions. Noting that in the interval $[T_-, 0]$, $\sigma(x,z(\tau)) = 0$ only at $\tau = \tau_{ret}$, and using $d\tau = \dot{\sigma}^{-1}d\sigma$, with $(\dot{\ }) = d/d\tau$, we have,

$$\Phi^{ret}(x) = q\left(\frac{U(x,z(\tau))}{\dot{\sigma}}\right)_{ret} - q\int_{T_-}^{\tau_{ret}(x)}V(x,z(\tau))d\tau + q\int_{-\infty}^{T_-}G^{ret}(x,z(\tau))d\tau. \qquad (2.2.4)$$

The gradient of Φ with respect to x is given by

$$\nabla_\alpha\Phi^{ret} = q\nabla_\alpha\left[\left(\frac{U}{\dot{\sigma}}\right)_{ret}\right] + qV\nabla_\alpha\tau_{ret} - q\int_{T_-}^{\tau_{ret}}\nabla_\alpha V d\tau + q\int_{-\infty}^{T_-}\nabla_\alpha G^{ret}d\tau. \qquad (2.2.5)$$

Because $\nabla_\alpha V(x,z(\tau))$ and $\nabla_\alpha G^{ret}(x,z(\tau))$ are vectors in the tangent space at x for all values of τ, the integrals are well defined.

Noticing that, for $T_- \leq \tau < \tau_{ret}$, $G^{ret}(x,z(\tau)) = -V(x,z(\tau))$, we write

$$\nabla_\alpha\Phi^{ret} = q\nabla_\alpha\left[\left(\frac{U}{\dot{\sigma}}\right)_{ret}\right] + qV\nabla_\alpha\tau_{ret} + q\lim_{h\to 0}\int_{-\infty}^{\tau_- - h}\nabla_\alpha G^{ret}d\tau. \qquad (2.2.6)$$

The retarded and advanced solutions to the solutions take the form

$$\Phi^{ret/adv} = q\left[\frac{U(x,z)}{\dot{\sigma}}\right]_{ret/adv} \pm q\lim_{h\to 0}\int_{\mp\infty}^{\tau_{ret/adv}\mp h}G^{ret/adv}(x,z)d\tau, \qquad (2.2.7)$$

and

$$\begin{aligned}
\nabla_\alpha\Phi^{ret/adv} &= q\nabla_\alpha\left[\frac{U(x,z)}{\dot{\sigma}}\right]_{ret/adv} \pm qV(x,z)\nabla_\alpha\tau_{ret/adv} \\
&\pm q\lim_{h\to 0}\int_{\mp\infty}^{\tau_{ret/adv}\mp h}\nabla_\alpha G^{ret/adv}(x,z)d\tau. \qquad (2.2.8)
\end{aligned}$$

Further progress is difficult with out obtaining expansions of the three bi-scalars, U, V, and σ.

Expanding the Biscalars, $U(x,z)$, $V(x,z)$, and $\sigma(x,z)$

The quantities $U(x,z)$ and $V(x,z)$ have the local expansions [37]

$$U(x,z) = 1 + \frac{1}{12} R_{\alpha'\beta'} \nabla^{\alpha'}\sigma(x,z)\nabla^{\beta'}\sigma(x,z) + O(\epsilon^3), \tag{2.2.9}$$

$$V(x,z) = -\frac{1}{12} R(z) + O(\epsilon), \tag{2.2.10}$$

where $\nabla^{\alpha'}$ is defined to be the contravariant derivative at the position of the particle (z), $R_{\alpha\beta}$ is the Ricci Tensor, and $R(z)$ is the Ricci Scalar.

Now it is necessary to express $\dot{\sigma}_{ret/adv}$ in terms of the coordinates $x^{\hat{\alpha}}$, and the particle's 4-velocity u^α, acceleration a^α, and jerk $\dot{a}^\alpha := u^\beta \nabla_\beta a^\alpha$ at $\tau = 0$. We will write $\dot{\sigma}_{ret/adv} = -(u^\alpha y_\alpha)_{ret/adv}$, where $-y_{\alpha,\,ret}$ and $-y_{\alpha,\,adv}$ are the gradients with respect to z of $\sigma(x,z)$ at $z_{ret} = z(\tau_{ret})$ and $z_{adv} = z(\tau_{adv})$,

$$y_{\alpha,ret/adv} := -\left(\nabla_\alpha \sigma\right)_{ret/adv}. \tag{2.2.11}$$

The contravariant vectors $y^\alpha_{ret/adv}$ are tangent to affinely parameterized null geodesics from $z(\tau_{ret/adv})$ to x. Solving the geodesic equation iteratively, produces

$$y^{\hat{\alpha}}_{ret} = \left(x^{\hat{\alpha}} - z^{\hat{\alpha}}_{ret}\right) - \frac{1}{3} R^{\hat{\alpha}}{}_{\hat{\mu}\hat{\nu}\hat{\gamma}} z^{\hat{\gamma}}_{ret} \left(x^{\hat{\mu}} - z^{\hat{\mu}}_{ret}\right)\left(x^{\hat{\nu}} - z^{\nu}_{ret}\right) + O(\epsilon^4). \tag{2.2.12}$$

For the advanced term, $y^{\hat{\alpha}}_{adv}$, replace each subscript "ret" by "adv". Next, expand $z^{\hat{\alpha}}(\tau)$ about $\tau = 0$:

$$z^{\hat{\alpha}}(\tau_{ret/adv}) = z^{\hat{\alpha}}(0) + \partial_\tau z^{\hat{\alpha}}\big|_{\tau=0} \tau_{ret/adv} + \frac{1}{2} \partial^2_\tau z^{\hat{\alpha}}\Big|_{\tau=0} \tau^2_{ret/adv} + O(\tau^3). \tag{2.2.13}$$

Using the form of the Christoffel symbols in RNC, $\Gamma^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}} = -\frac{2}{3} R^{\hat{\alpha}}{}_{(\hat{\beta}\hat{\gamma})\hat{\delta}} x^{\hat{\delta}}$, and the index symmetries of the Riemann tensor gives

$$a^{\hat{\alpha}} = u^{\hat{\beta}} \nabla_{\hat{\beta}} u^{\hat{\alpha}}|_{\tau=0} = \partial^2_\tau z^{\hat{\alpha}}|_{\tau=0}, \qquad \dot{a}^{\hat{\alpha}} = u^{\hat{\beta}} \nabla_{\hat{\beta}} a^{\hat{\alpha}}|_{\tau=0} = \partial^3_\tau \hat{z}^{\alpha}|_{\tau=0}, \tag{2.2.14}$$

whence

$$z^{\hat{\alpha}}(t_{ret/adv}) = u^{\hat{\alpha}} \tau_{ret} + \frac{1}{2} a^{\hat{\alpha}} \tau^2_{ret} + \frac{1}{6} \dot{a}^{\hat{\alpha}} \tau^3_{ret} + O(\tau^4_{ret}), \tag{2.2.15}$$

with each coefficient evaluated at $\tau = 0$. Now we use the relation $(g_{\alpha\beta} y^\alpha y^\beta)_{ret/adv} = 0$ to find $\tau_{ret/adv}$ in terms of $u^{\hat{\alpha}}$ and $x^{\hat{\alpha}}$. Writing $\tau_{ret/adv} = \tau_1 + \tau_2 + O(\tau^3)$, with $\tau_n = O(\epsilon^n)$ yields

$$\tau_1 = -\left(u_{\hat{\alpha}} x^{\hat{\alpha}} \pm \sqrt{\left(\eta_{\hat{\alpha}\hat{\beta}} + u_{\hat{\alpha}} u_{\hat{\beta}}\right) x^{\hat{\alpha}} x^{\hat{\beta}}}\right), \tag{2.2.16}$$

where the $\pm$ corresponds to retarded (+) and advanced (-) solutions and $u_{\hat{\alpha}}$ is evaluated at $\tau = 0$. Let

$$q_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_\beta \tag{2.2.17}$$

be the projection operator orthogonal to u_α and, with notation motivated by Eq. (2.2.24) below, write $\hat{S}_0 = q_{\hat{\alpha}\hat{\beta}} x^{\hat{\alpha}} x^{\hat{\beta}}$, where $q_{\hat{\alpha}\hat{\beta}}$ is evaluated at $z(0)$. Then

$$\tau_1 = -\left(u_{\hat{\mu}} x^{\hat{\mu}} \pm \sqrt{\hat{S}_0} \right). \tag{2.2.18}$$

Similarly,

$$\tau_2 = \pm \frac{a_{\hat{\alpha}} x^{\hat{\alpha}}}{2\sqrt{\hat{S}_0}}\, \tau_1^2. \tag{2.2.19}$$

Finally, substituting Eqs. (2.2.16), and (2.2.19) into Eq. (2.2.15)provides an expression for $z^{\hat{\alpha}}_{ret/adv}$ (and thus $y^{\hat{\alpha}}$) entirely in terms of $x^{\hat{\alpha}}$ and of $u^{\hat{\alpha}}$ and their derivatives at $t = 0$.

The next step is to expand $\dot{\sigma}$ about $\epsilon = 0$. To do this, we focus on $\dot{\sigma}^2$ and pattern the calculation on that of [21]. Thus, we write

$$\dot{\sigma}^2_{ret/adv} = (u_{\hat{\alpha}} y^{\hat{\alpha}})^2_{ret/adv} = \left(q_{\hat{\alpha}\hat{\beta}} y^{\hat{\alpha}} y^{\hat{\beta}} \right)_{ret/adv}. \tag{2.2.20}$$

Here u^α is the four velocity of the particle at the retarded or advanced times (we treat this in a similar manner to the way we treated $z^{\hat{\alpha}}_{ret/adv}$, using a similar expansion as in Eq. (2.2.15)). Since $y^\alpha_{ret/adv}$ is a null vector, we can add the term $g_{\hat{\alpha}\hat{\beta}} y^{\hat{\alpha}} y^{\hat{\beta}} = 0$. The reason for this change will soon be clear.

To keep track of the relevant terms in the calculation, we borrow a term from [21], and generalize it. Define $\hat{S}$ as [4]

$$\hat{S} := \left[q_{\hat{\alpha}\hat{\beta}} (x^{\hat{\alpha}} - z^{\hat{\alpha}})(x^{\hat{\beta}} - z^{\hat{\beta}}) \right]_{ret/adv}. \tag{2.2.21}$$

This definition leads to the expression

$$\dot{\sigma}^2_{ret/adv} = S_{ret/adv} + \frac{1}{3} R_{\hat{\alpha}\hat{\gamma}\hat{\beta}\hat{\lambda}} x^{\hat{\alpha}} x^{\hat{\beta}} u^{\hat{\gamma}} u^{\hat{\lambda}} (x^{\hat{\iota}} x_{\hat{\iota}}) + O(\epsilon^5). \tag{2.2.22}$$

Here and in the rest of this section, $q_{\hat{\alpha}\hat{\beta}}$, $u^{\hat{\alpha}}$, $a^{\hat{\alpha}}$, and $\dot{a}^{\hat{\alpha}}$ will all be assumed to be evaluated at $\tau = 0$. When we expand S about $\epsilon = 0$, we find

$$\hat{S} = \hat{S}_0 + \hat{S}_1 + \hat{S}_2 + ... \tag{2.2.23}$$

[4]It is useful to note that in [21] the use of the hat denoted a quantity evaluated at $\delta r = 0$, whereas we use hats to specify that the expression is one found using RNCs. When we need to make a similar evaluation we will denote these quantities with a tilde.

where $\hat{S}_n = O(\epsilon^{n+2})$. Explicitly, we have

$$\hat{S}_0 = (\eta_{\hat{\alpha}\hat{\beta}} + u_{\hat{\alpha}}u_{\hat{\beta}})x^{\hat{\alpha}}x^{\hat{\beta}}, \tag{2.2.24}$$

$$\hat{S}_1 = \eta_{\hat{\alpha}\hat{\beta}}a_{\hat{\gamma}}x^{\hat{\alpha}}x^{\hat{\beta}}x^{\hat{\gamma}}, \tag{2.2.25}$$

and

$$\hat{S}_2 = S_2^{(1)} \pm S_2^{(\pm)} = \left[\Sigma^{(1)}_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\lambda}} \pm \frac{x^{\hat{\delta}}}{\sqrt{\hat{S}_0}}\Sigma^{(\pm)}_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\lambda}\hat{\delta}}\right] x^{\hat{\alpha}}x^{\hat{\beta}}x^{\hat{\gamma}}x^{\hat{\lambda}}, \tag{2.2.26}$$

where the quantities $\Sigma^{(1)}_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\lambda}}$ and $\Sigma^{(\pm)}_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\lambda}\hat{\delta}}$ in Eq. (2.2.26) are

$$\Sigma^{(1)}_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\lambda}} := \frac{a^2}{12}q_{\hat{\alpha}\hat{\beta}}\left((\eta_{\hat{\gamma}\hat{\lambda}} + 7u_{\hat{\gamma}}u_{\hat{\lambda}}) - u_{\hat{\alpha}}u_{\hat{\beta}}u_{\hat{\gamma}}u_{\hat{\lambda}}\right) - \frac{u_{\hat{\lambda}}\dot{a}_{\hat{\gamma}}}{3}(3\eta_{\hat{\alpha}\hat{\beta}} + 2u_{\hat{\alpha}}u_{\hat{\beta}}) \tag{2.2.27}$$

and

$$\Sigma^{(\pm)}_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\lambda}\hat{\delta}} := \frac{2}{3}(\eta_{\hat{\alpha}\hat{\beta}} + u_{\hat{\alpha}}u_{\hat{\beta}})(\eta_{\hat{\gamma}\hat{\lambda}} + u_{\hat{\gamma}}u_{\hat{\lambda}})(a^2u_{\hat{\delta}} - \dot{a}_{\hat{\delta}}). \tag{2.2.28}$$

It is also useful to define

$$r_{\hat{\alpha}} := \frac{1}{2}\nabla_{\hat{\alpha}}\hat{S}_0 = \nabla_{\hat{\alpha}}\left(\eta_{\hat{\mu}\hat{\nu}} + u_{\hat{\mu}}u_{\hat{\nu}}\right)x^{\hat{\mu}}x^{\hat{\nu}} = \left(\eta_{\hat{\alpha}\hat{\mu}} + u_{\hat{\mu}}u_{\hat{\alpha}}\right)x^{\hat{\mu}}. \tag{2.2.29}$$

We now have the information to write the expansion of the first term in Eq. (2.2.7) (sometimes called the 'direct' term). We use Eqs. (2.2.9), (2.2.22), (2.2.23), (2.2.24), (2.2.25), and (2.2.26) to expand $\Phi^{ret/adv}$ to the first three orders in ϵ:

$$\begin{aligned}\Phi^{ret/adv} \quad = \quad & \frac{q}{\sqrt{\hat{S}_0}}\left[1 - \frac{\hat{S}_1}{2\hat{S}_0} + \frac{3}{8}\left(\frac{\hat{S}_1}{\hat{S}_0}\right)^2 - \frac{\hat{S}_2}{2\hat{S}_0}\right] - \frac{q}{6\hat{S}_0^{3/2}}R_{\hat{\alpha}\hat{\gamma}\hat{\beta}\hat{\lambda}}u^{\hat{\lambda}}u^{\hat{\gamma}}x^{\hat{\alpha}}x^{\hat{\beta}}x^2 \\ & + \frac{qR_{\hat{\alpha}\hat{\beta}}}{12}\left[\frac{r^{\hat{\alpha}}r^{\hat{\beta}} + S_0u^{\hat{\alpha}}u^{\hat{\beta}}}{\sqrt{\hat{S}_0}} \pm 2(x^{\hat{\alpha}}u^{\hat{\beta}} + u^{\hat{\alpha}}u^{\hat{\beta}}u_{\hat{\gamma}}x^{\hat{\gamma}})\right] \\ & \pm q\lim_{h\to 0}\int_{\mp\infty}^{\tau_{ret/adv}\mp h} G^{ret/adv}(x,z)d\tau + O(\epsilon^2),\end{aligned} \tag{2.2.30}$$

where $x^2 \equiv x^{\hat{\epsilon}}x_{\hat{\epsilon}}$.

It is instructive to see Eq. (2.2.30) written in terms of the acceleration and jerk. Using

Eqs. (2.2.24)-(2.2.28), we obtain

$$\begin{aligned}\Phi^{ret/adv} &= \frac{q}{\sqrt{q_{\hat{\mu}\hat{\nu}}x^{\hat{\mu}}x^{\hat{\nu}}}}\left[1-\frac{a_{\hat{\gamma}}x^{\hat{\gamma}}x^2}{2q_{\hat{\mu}\hat{\nu}}x^{\hat{\mu}}x^{\hat{\nu}}}\left(1-\frac{3}{4}\frac{a_{\hat{\gamma}}x^{\hat{\gamma}}x^2}{q_{\hat{\mu}\hat{\nu}}x^{\hat{\mu}}x^{\hat{\nu}}}\right)\right]\mp\frac{q}{3}x^{\hat{\delta}}(a^2u_{\hat{\delta}}-\dot{a}_{\hat{\delta}}) \\ &\quad -\frac{qa^2\left(q_{\hat{\alpha}\hat{\beta}}(\eta_{\hat{\gamma}\hat{\lambda}}+7u_{\hat{\gamma}}u_{\hat{\lambda}})-u_{\hat{\alpha}}u_{\hat{\beta}}u_{\hat{\gamma}}u_{\hat{\lambda}}\right)}{24\left(q_{\hat{\mu}\hat{\nu}}x^{\hat{\mu}}x^{\hat{\nu}}\right)^{3/2}}x^{\hat{\alpha}}x^{\hat{\beta}}x^{\hat{\gamma}}x^{\hat{\lambda}} \\ &\quad -\frac{4qu_{\hat{\gamma}}a_{\hat{\lambda}}(3\eta_{\hat{\alpha}\hat{\beta}}+2u_{\hat{\alpha}}u_{\hat{\beta}})}{24\left(q_{\hat{\mu}\hat{\nu}}x^{\hat{\mu}}x^{\hat{\nu}}\right)^{3/2}}x^{\hat{\alpha}}x^{\hat{\beta}}x^{\hat{\gamma}}x^{\hat{\lambda}}-\frac{qR_{\hat{\alpha}\hat{\gamma}\hat{\beta}\hat{\lambda}}u^{\hat{\lambda}}u^{\hat{\gamma}}x^{\hat{\alpha}}x^{\hat{\beta}}x^2}{6\left(q_{\hat{\mu}\hat{\nu}}x^{\hat{\mu}}x^{\hat{\nu}}\right)^{3/2}} \\ &\quad +\frac{qR_{\hat{\alpha}\hat{\beta}}}{12}\left[\frac{x^{\hat{\alpha}}x^{\hat{\beta}}+2u^{\hat{\alpha}}x^{\hat{\beta}}(u_{\hat{\gamma}}x^{\hat{\gamma}})+u^{\hat{\alpha}}u^{\hat{\beta}}(x^{\epsilon}x_{\epsilon}+2(u_{\hat{\gamma}}x^{\hat{\gamma}})^2)}{\sqrt{q_{\hat{\mu}\hat{\nu}}x^{\hat{\mu}}x^{\hat{\nu}}}}\right] \\ &\quad \pm\frac{qR_{\hat{\alpha}\hat{\beta}}}{6}(x^{\hat{\alpha}}u^{\hat{\beta}}+u^{\hat{\alpha}}u^{\hat{\beta}}u_{\hat{\gamma}}x^{\hat{\gamma}}) \\ &\quad \pm q\lim_{h\to 0}\int_{\mp\infty}^{\tau_{ret/adv}\mp h}G^{ret/adv}(x,z)d\tau+O(\epsilon^2).\end{aligned} \tag{2.2.31}$$

Noting that $\hat{S}_0 = r_{\hat{\alpha}}r^{\hat{\alpha}}$, we write Eq. (2.2.31) as

$$\begin{aligned}\Phi^{ret/adv} &= \frac{q}{r}\left[1-\frac{a_{\hat{\gamma}}r^{\hat{\gamma}}x^2}{2r^2}-\frac{1}{2r^2}\left(\frac{a^2}{12}\left(r^4+6r^2(u_{\hat{\alpha}}x^{\hat{\alpha}})^2-(u_{\hat{\alpha}}x^{\hat{\alpha}})^4\right)\right)\right. \\ &\quad \left.+\frac{3}{8}\left(\frac{a_{\hat{\gamma}}r^{\hat{\gamma}}x^2}{r^2}\right)^2\right] \\ &\quad -\frac{q}{12r^3}\left[2u_{\hat{\gamma}}x^{\hat{\gamma}}a_{\hat{\mu}}r^{\hat{\mu}}\left(3r^2-(u_{\hat{\sigma}}x^{\hat{\sigma}})^2\right)+2R_{\hat{\alpha}\hat{\gamma}\hat{\beta}\hat{\delta}}x^{\hat{\alpha}}x^{\hat{\beta}}u^{\hat{\gamma}}u^{\hat{\delta}}x^2\right. \\ &\quad \left.-r^2R_{\hat{\alpha}\hat{\beta}}\left(r^{\hat{\alpha}}r^{\hat{\beta}}+r^2u^{\hat{\alpha}}u^{\hat{\beta}}\right)\right]\pm\frac{q}{6}\left[R_{\hat{\alpha}\hat{\beta}}r^{\hat{\alpha}}u^{\hat{\beta}}+2x^{\hat{\alpha}}(\dot{a}_{\hat{\alpha}}-a^2u_{\hat{\alpha}})\right] \\ &\quad \pm q\lim_{h\to 0}\int_{\mp\infty}^{\tau_{ret/adv}\mp h}G^{ret/adv}(x,z)d\tau+O(\epsilon^2).\end{aligned} \tag{2.2.32}$$

Therefore, using Eq. (2.2.8), we can write the gradient of the retarded and advanced fields as

$$\begin{aligned}\nabla_\alpha\Phi^{ret/adv} &= \nabla_\alpha\left[\left(\frac{qU(x,z)}{\dot{\sigma}}\right)_{ret/adv}\right]-\frac{R(z)q}{12}\left(\frac{\nabla_\alpha\hat{S}_0}{2\sqrt{\hat{S}_0}}\pm u_\alpha\right) \\ &\quad \pm q\nabla_\alpha\lim_{h\to 0}\int_{\mp\infty}^{\tau_{ret/adv}\mp h}G^{ret/adv}(x,z)d\tau.\end{aligned} \tag{2.2.33}$$

Writing out the gradient of the scalar field in terms of the S_n's leads to

$$\begin{aligned}
\nabla_{\hat{\alpha}}\left[\Phi^{ret/adv}\right] = \; & q\left[-\frac{\nabla_{\hat{\alpha}}\hat{S}_0}{2\hat{S}_0^{3/2}} - \frac{1}{2}\left(\frac{\nabla_{\hat{\alpha}}\hat{S}_1}{\hat{S}_0^{3/2}} - \frac{3}{2}\frac{\hat{S}_1\nabla_{\hat{\alpha}}\hat{S}_0}{\hat{S}_0^{5/2}}\right) - \frac{15}{16}\frac{\hat{S}_1^2\nabla_{\hat{\alpha}}\hat{S}_0}{\hat{S}_0^{7/2}}\right. \\
& \left. +\frac{3}{4}\frac{\hat{S}_1\nabla_{\hat{\alpha}}\hat{S}_1}{\hat{S}_0^{5/2}}\right] + q\left[-\frac{1}{2}\left(\frac{\nabla_{\hat{\alpha}}\hat{S}_2}{\hat{S}_0^{3/2}} - \frac{3}{2}\frac{\hat{S}_2\nabla_{\hat{\alpha}}\hat{S}_0}{\hat{S}_0^{5/2}}\right)\right. \\
& \left.\pm\frac{1}{6}R_{\hat{\mu}\hat{\beta}}u^{\hat{\beta}}\left(\delta^{\hat{\mu}}{}_{\hat{\alpha}} + u^{\hat{\mu}}u_{\hat{\alpha}}\right)\right] - \frac{qR_{\hat{\mu}\hat{\nu}}}{24}\left[\frac{\nabla_{\hat{\alpha}}\hat{S}_0}{\hat{S}_0^{3/2}}\left(r^{\hat{\mu}}r^{\hat{\nu}} + \hat{S}_0u^{\hat{\mu}}u^{\hat{\nu}}\right)\right. \\
& \left. -\frac{2}{\sqrt{\hat{S}_0}}\left(r^{\hat{\mu}}\left(\delta^{\hat{\nu}}_{\hat{\alpha}} + u^{\hat{\nu}}u_{\hat{\alpha}}\right) + u^{\hat{\mu}}u^{\hat{\nu}}\nabla_{\hat{\alpha}}\hat{S}_0\right)\right] - \frac{qR(z)}{12}\left(\frac{\nabla_{\hat{\alpha}}\hat{S}_0}{2\sqrt{\hat{S}_0}} \pm u_{\hat{\alpha}}\right) \\
& -\frac{qR_{\hat{\mu}\hat{\gamma}\hat{\nu}\hat{\delta}}u^{\hat{\gamma}}u^{\hat{\delta}}x^{\hat{\mu}}}{12\hat{S}_0^{5/2}}\left(4\hat{S}_0x^2\delta^{\hat{\nu}}_{\hat{\alpha}} + 4\hat{S}_0x^{\hat{\nu}}x_{\hat{\alpha}} - 3x^{\hat{\nu}}x^2\nabla_{\hat{\alpha}}\hat{S}_0\right) \\
& \pm q\nabla_{\hat{\alpha}}\lim_{h\to 0}\int_{\mp\infty}^{\tau_{ret/adv}\mp h} G^{ret/adv}(x,z)d\tau + O(\epsilon^2). \qquad (2.2.34)
\end{aligned}$$

Thus, we have reached an expression for the derivative of the retarded and advanced fields due to an accelerated particle moving through an arbitrary curved spacetime. This equation will form the basis of our understanding of renormalization, and in the following sections we will come to understand that this hideous looking equation is in fact much simpler than its initial appearance.

Some important aspects of Eq. (2.2.34), become more apparent when re-expressed in terms of $a^\mu, \dot{a}^\mu$, and r^μ, $\nabla_{\hat{\alpha}}\Phi^{ret/adv}$;

$$\begin{aligned}
\nabla_{\hat{\alpha}}\Phi^{ret/adv} &= q\Bigg[-\frac{r_{\hat{\alpha}}}{r^3} - \frac{1}{2}\left(\frac{a_{\hat{\alpha}}x^2 + 2a_{\hat{\gamma}}x^{\hat{\gamma}}x_{\hat{\alpha}}}{r^3} - 3\frac{a_{\hat{\gamma}}r^{\hat{\gamma}}x^2 r_{\hat{\alpha}}}{r^5}\right) \\
&+\frac{3}{4}\frac{a_{\hat{\gamma}}r^{\hat{\gamma}}x^2(a_{\hat{\alpha}}x^2 + 2a_{\hat{\gamma}}x^{\hat{\gamma}}x_{\hat{\alpha}})}{r^5}\Bigg] + q\left[-\frac{15}{8}\frac{\left(a_{\hat{\gamma}}r^{\hat{\gamma}}x^2\right)^2 r_{\hat{\alpha}}}{r^7}\right] \\
&-q\Bigg[\frac{a^2}{24r^5}\Big(r^4 r_{\hat{\alpha}} + 12r^4 u_{\hat{\gamma}}x^{\hat{\gamma}}u_{\hat{\alpha}} - 6r^2\left(u_{\hat{\gamma}}x^{\hat{\gamma}}\right)^2 r_{\hat{\alpha}} - 4r^2\left(u_{\hat{\gamma}}x^{\hat{\gamma}}\right)^3 u_{\hat{\alpha}} \\
&+3\left(u_{\hat{\gamma}}x^{\hat{\gamma}}\right)^4 r_{\hat{\alpha}}\Big)\Bigg] - \frac{q}{2}\Bigg[\frac{2\dot{a}_{\hat{\beta}}x^{\hat{\beta}}\left(u_{\hat{\gamma}}x^{\hat{\gamma}}\right)^2}{3r^5}\left(r^2 u_{\hat{\alpha}} - r_{\hat{\alpha}}u_{\hat{\gamma}}x^{\hat{\gamma}}\right) + \\
&\left(1 - \frac{1}{3}\left(\frac{u_{\hat{\gamma}}x^{\hat{\gamma}}}{r}\right)^2\right)\frac{1}{r^3}\left(u_{\hat{\gamma}}x^{\hat{\gamma}}\dot{a}_{\hat{\beta}}x^{\hat{\beta}}r_{\hat{\alpha}} - r^2\left(u_{\hat{\alpha}}\dot{a}_{\hat{\beta}}x^{\hat{\beta}} + \dot{a}_{\hat{\alpha}}u_{\hat{\beta}}x^{\hat{\beta}}\right)\right)\Bigg] \\
&+\frac{qR_{\hat{\mu}\hat{\nu}}}{12}\left[\frac{1}{r}\left(r^{\hat{\mu}}\left(\delta^{\hat{\nu}}_{\hat{\alpha}} + u^{\hat{\nu}}u_{\hat{\alpha}}\right) + 2u^{\hat{\mu}}u^{\hat{\nu}}r_{\hat{\alpha}}\right) - \frac{r_{\hat{\alpha}}}{r^3}\left(r^{\hat{\mu}}r^{\hat{\nu}} + r^2 u^{\hat{\mu}}u^{\hat{\nu}}\right)\right] \\
&-\frac{qR_{\hat{\mu}\hat{\gamma}\hat{\nu}\hat{\delta}}u^{\hat{\gamma}}u^{\hat{\delta}}x^{\hat{\mu}}}{6r^5}\left(2r^2x^2\delta^{\hat{\nu}}_{\hat{\alpha}} + 2r^2x^{\hat{\nu}}x_{\hat{\alpha}} - 3x^{\hat{\nu}}x^2 r_{\hat{\alpha}}\right) - \frac{qR(z)}{12}\left(\frac{r_{\hat{\alpha}}}{r}\right) \\
&\pm\frac{q}{12}\left[4\left(\dot{a}_{\hat{\alpha}} - a^2u_{\hat{\alpha}}\right) + 2R_{\hat{\mu}\hat{\beta}}u^{\hat{\beta}}\left(\delta^{\hat{\mu}}_{\hat{\alpha}} + u^{\hat{\mu}}u_{\hat{\alpha}}\right) - R(z)u_{\hat{\alpha}}\right] \\
&\pm q\lim_{h\to 0}\int_{\mp\infty}^{\tau_{ret/adv}\mp h}\nabla_{\hat{\alpha}}G^{ret/adv}(x,z)d\tau + O(\epsilon^1).
\end{aligned} \tag{2.3.1}$$

While a cursory glance at Eq. (2.3.1)[5] is unlikely to provide any illumination, this equation contains a wealth of information. Let us stop to consider only the field that would exist in flat spacetime. This would include the first three lines, and the first term of the second line (the integral term, known as the "tail", vanishes in flat spacetime because the flat spacetime Green's function only has support on the light cones).

The first term is clearly the inverse square law from a coulomb field. The second, which also diverges, is proportional to the acceleration. The rest of the terms in the first five lines all share a common feature: they have an *odd number of unit normal vectors.* This means that if we consider any one of these terms and take the limit as we approach

[5] This is a more general expression than is given in Quinn [18]. Only when the field point x is chosen to be along a geodesic orthogonal to the trajectory at $z(0)$ (that is, when $u_{\hat{\alpha}}x^{\hat{\alpha}} = 0$) does this match Quinn's expression.

the particle from one direction, we will get the negative of the value we would get if we approached it in the opposite way. That is to say that these terms, while not divergent, do not give a well defined field at the position of the particle.

The second thing to notice from the flat spacetime field is that the $\pm$ term corresponds to the $\dddot{x}$ force noted by Dirac. Therefore, any prescription that we make to find the self-force needs to eliminate all of the divergent or direction dependent terms while leaving the term that actually produce the self-force.

With this material we can now understand the Quinn (Quinn-Wald) axioms for the scalar (electric and gravitational) self-force(s) [18] ([20]).

Quinn's first axiom, the comparison axiom can be stated as follows:

> Consider two point particles in two possibly different spacetimes, each particle having scalar charge q. Suppose that, at points $z(0)$ and $\tilde{z}(0)$ on their respective trajectories, the magnitude of the particles' 4-accelerations coincide. We may then choose RNC systems about $z(0)$ and about $\tilde{z}(0)$ for which the components of the 4-velocities and 4-accelerations coincide:
>
> $$u^{\hat{\alpha}} = \tilde{u}^{\hat{\alpha}}, \qquad a^{\hat{\alpha}} = \tilde{a}^{\hat{\alpha}}. \tag{2.3.2}$$
>
> Let Φ and $\tilde{\Phi}$ be the retarded scalar fields of the particles. With the RNC systems used to identify neighborhoods of $z(0)$ and $\tilde{z}(0)$, the difference between the renormalized scalar forces, $f^R_{Q\alpha}$ and $\tilde{f}^R_{Q\alpha}$ is given by the limit as $r \to 0$ of the gradients of the fields averaged over a sphere of geodesic distance r about $z(0)$.[6]

$$f_Q^{R,\hat{\alpha}} - \tilde{f}_Q^{R,\hat{\alpha}} = q \lim_{r\to 0} \langle \nabla^{\hat{\alpha}}\Phi - \nabla^{\hat{\alpha}}\tilde{\Phi}\rangle_r. \tag{2.3.3}$$

Quinn's second axiom simply states that the renormalized scalar force vanishes for the half-advanced + half-retarded field of a uniformly accelerated charge in flat space:

> If, for a uniformly accelerated scalar charge in flat space, $\tilde{\Phi} = \frac{1}{2}(\tilde{\Phi}^{ret} + \tilde{\Phi}^{adv})$, then $\tilde{f}_Q^{R,\alpha} = 0$.

[6] With S_r the set of points that lie a geodesic distance r from $z(0)$ along a geodesic perpendicular to the trajectory, the average of a function f is $\langle f\rangle_r := |S_r|^{-1}\int_{S_r} f dS$, where $|S_r|$ is the area of S_r.

To define the self-force, we assume that the spacetime of the field Φ is globally hyperbolic so that retarded and advanced fields are well defined, and we set $\Phi = \Phi^{ret}$. With this restriction, the axioms imply that the self-force is given by

$$f^{R,\hat{\alpha}} = q \lim_{r\to 0} \langle \nabla^{\hat{\alpha}} \Phi^{ret} - \nabla^{\hat{\alpha}} \tilde{\Phi} \rangle_r. \tag{2.3.4}$$

As in this equation, we will henceforth use the RNC identification of normal neighborhoods of the flat and curved spacetimes to regard $\tilde{\Phi}$ as a field on C.

For ease of comparison, we will rewrite history slightly and introduce some terminology that only came into usage after Quinn, and was formally defined only later by Detweiler and Whiting [36]. So far, we have discussed the retarded and advanced forces, $f^{ret}_\alpha, f^{adv}_\alpha$ and the renormalized force f^R_α. We will now introduce the concept of a *singular force* f^S_α which is the force due to the singular field Φ^S. The singular field is just the field that contains the singular structure of the retarded field. The singular field does not have to be uniquely defines, although as we will see in the next section, there are certainly some definitions that are more useful than others. For us, we will say that the singular field as described by Quinn is the half-advanced-half-retarded flat spacetime field, and therefore,

$$\begin{aligned} f^{R,\hat{\alpha}} &= q \lim_{r\to 0} \langle \nabla^{\hat{\alpha}} \Phi^{ret} - \nabla^{\hat{\alpha}} \Phi^S_Q \rangle_r & (2.3.5) \\ &= q \lim_{r\to 0} \langle f^{ret,\hat{\alpha}} - f^{S,\hat{\alpha}}_Q \rangle_r, & (2.3.6) \end{aligned}$$

where we use the subscript Q to denote that these are the singular quantities effectively used by Quinn.

This is an elegant procedure, and provided the crucial first step in understanding how to renormalize the self-force. The angle-average is very useful conceptually[7], but for practical applications it can be quite cumbersome.

For example, in most cases, the only clear-cut way of generating the solutions for the retarded field is to express them as modes of angular harmonic functions (typically spherical harmonics or spheroidal harmonic), using a coordinate basis with the origin at the central singularity of the black hole. The angle average here is an angle average

[7]one can think of this angle average as merely saying that the total force felt by the particle is the sum of all of the forces on it, with the angle average acting to enforce the summation; it adds the force from above to the force from below, the force from the right to the force from the left, etc.

about the particle, an angle average that would be difficult to perform in any practical calculation using the angular harmonics.

2.4 Detweiler and Whiting's Singular and Renormalized Fields

Detweiler and Whiting [36] sought a more practical renormalization routine than that proposed by *MiSaTaQuWa*. Using their method, we seek to define a renormalized field, ϕ^R, that is defined in the normal neighborhood of the particle, and is smooth in the entire domain, even at the particle. It is this renormalized field which determines the motion of the particle itself.

To understand the motivation for their definitions, look again at Eq. (2.3.1). Before, we explained how the half-advanced-half retarded flat spacetime field would include all of the terms that either diverge or are direction dependent at the particle. If we consider the curved spacetime fields, we notice that once again, all of the terms in $\nabla_\alpha \phi$ that are shared between the advanced and retarded solutions would fit in this description. As such, it would be tempting to simply say that the singular field, ϕ^S, which we must subtract from the retarded field to generate ϕ^R would be

$$\phi^S_{(1)} = \frac{1}{2}\left[\phi^{ret} + \phi^{adv}\right], \tag{2.4.1}$$

where the subscript (1) indicates that this is our first guess at the singular field. There is, however, a flaw to this definition–the singular field represents the behavior of the retarded solution very close to the particle, and $\phi^S_{(1)}$ includes contributions form the tail terms, which include contributions from the entire history of the particle (in fact both past and future history because this definition includes both the advanced and retarded solutions).

In order to overcome this objection, a natural second attempt would be to define the singular field as

$$\phi^S_{(2)} = \frac{1}{2}\left[\frac{U}{\dot\sigma}\right]_{ret} + \frac{1}{2}\left[\frac{U}{\dot\sigma}\right]_{adv}. \tag{2.4.2}$$

This term still does not quite suffice, however, and to understand why, let us return to Eq. (2.2.33), reproduced below

$$\nabla_\alpha \Phi^{ret/adv} = \nabla_\alpha \left[\left(\frac{qU(x,z)}{\dot{\sigma}} \right)_{ret/adv} \right] - \frac{R(z)q}{12} \left(\frac{\nabla_\alpha \hat{S}_0}{2\sqrt{\hat{S}_0}} \pm u_\alpha \right)$$
$$\pm q \nabla_\alpha \lim_{h \to 0} \int_{\mp\infty}^{\tau_{ret/adv} \mp h} G^{ret/adv}(x,z) d\tau.$$

The second guess, while it does not include contributions from the entire history of the particle, it does not include contributions from *enough* of the history of the particle– it does not include the effects from the derivatives of the limits of the integral term.

To avoid these issues Detweiler and Whiting defined their singular field as

$$\Phi^S_{DW} = \frac{1}{2} \left[\left(\frac{U(x,z)}{\dot{\sigma}} \right)_{ret} + \left(\frac{U(x,z)}{\dot{\sigma}} \right)_{adv} \right] + \frac{q}{2} \int_{\tau_{ret}}^{\tau_{adv}} V(x,z) d\tau. \quad (2.4.3)$$

If we take the derivative of this field, then we would recover every term from Eq. (2.3.1) that is shared between the advanced and retarded fields– that is to say we recover every term that is either divergent or would produce a direction dependent term to the force.

Therefore, to get the renormalized field ϕ^R, we can simply write

$$\Phi^R_{DW} = \Phi^{ret} - \Phi^S_{DW}, \quad (2.4.4)$$

and the renormalized self-force is simply given by $f^R_\alpha = q\nabla_\alpha \Phi^R$. This means that we have a method for producing the renormalized self-force that does not include angle-averages about the particle, giving us a practical renormalization scheme.

Because the Detweiler-Whiting singular field is so central to the progress in self-force computation, it is worth pausing to enumerate some of the properties of the various fields defined in Eq. (2.4.4).

The field Φ^S is defined only locally, and in this region is a solution to $\nabla_\mu \nabla^\mu \Phi^S = -4\pi\rho$. As such, in the limit that the distance ϵ between the field point and the particle's position approaches zero, this field mimics the behavior of the retarded field and is dominated by the Coulomb, ϵ^{-1} field. If I take another field $\Psi \neq \Phi^S$ that also is a solution to the sourced field equations, then it can also be a singular field if

$$\nabla_\alpha \Phi^S_{DW} - \nabla_\alpha \Psi = 0. \quad (2.4.5)$$

That is to say, that the singular field is *not uniquely defined* and if I have one singular field, I can generate another singular field by adding to it a solution to the source-free equation which produces no force at the particle's position. On the other hand, following the prescription given by Detweiler and Whiting in Eq. (2.4.3), then there is no ambiguity. This distinction is crucial to understanding the application of the angle-average scheme discussed in the next section as it applies to the electrovac calculation in Chapter 4.

Because of this ambiguity in defining Φ^S, Φ^R suffers from the same ambiguity, although, once again, this is not ambiguous at all if one follows the Detweiler and Whiting prescription. By applying $\nabla_\alpha \nabla^\alpha$ to Eq. (2.4.4), it is clear that Φ^R is a solution to the source-free field equations as

$$\begin{aligned} \nabla_\alpha \nabla^\alpha \Phi^R_{DW} &= \nabla_\alpha \nabla^\alpha (\Phi^{ret} - \Phi^S_{DW}) \\ &= -4\pi\rho - (-4\pi\rho) = 0. \end{aligned} \tag{2.4.6}$$

While ambiguous definitions are typically to be avoided, this ambiguity is quite useful, because it means that we have some freedom to in choosing our singular field so as to give ϕ^R different properties. In particular, we will use this freedom to state that the mode-sum decomposition of the renormalized field evaluated at the particle falls off faster than any power of ℓ, a trait of C^∞ functions (see next Chapter).

2.4.1 The Interpretation for Gravity

The gravitational self-force can raise a host of very subtle questions. Perhaps the most important of these is the following: In general relativity, gravity is not considered to be a force, so, how can there be a gravitational self-force?

I waited to bring this up until now because we need Detweiler and Whiting's insights to conquer this question. First, let us consider the scalar self-force. Assume that I place a swarm of test particles near my scalar charge. Each of these test particles would experience a force given by the derivative of the *retarded field* of our point charge. The point charge itself will experience a force not due to its own retarded field but instead due to the renormalized field. Therefore, it experiences a very different force than the test particles nearby would experience.

When we consider gravitational perturbations, the metric perturbation $h_{\alpha\beta}$ takes the place of Φ. Let us assume that at a given instant, $t = 0$, the particle is traveling tangent to a geodesic of the background spacetime, and once again, consider a nearby test mass. The test mass would move along a geodesic of the total metric $g_{\alpha\beta} = g^0_{\alpha\beta} + h^{ret}_{\alpha\beta}$ (where $g^0_{\alpha\beta}$ is the unperturbed metric). The point mass producing the perturbation however, would instead move along a geodesic of the spacetime described by the metric $g_{\alpha\beta} = g^0_{\alpha\beta} + h^R_{\alpha\beta}$.

Therefore, the point particle is moving through a different spacetime, and since the geodesics of this spacetime do not necessarily match those of the background spacetime (or, for that matter, the metric of the spacetime a nearby particle would experience), it is said to experience a force, and this force is produced by the particle's interaction with its own gravitational field, and so we can describe this as the gravitational self-force.

2.4.2 Gralla's angle-average prescription

If we return to Eq. (2.3.1), and consider the case when the particle is moving on a geodesic. In this case, Gralla noticed that one could renormalize purely by angle averaging [23], and he utilized this to extend the ability to regularize the gravitational self-force to a wide range of gauges.

If we include the acceleration terms, however, this prescription would miss the terms proportional to the acceleration, terms which diverge as ϵ^{-1} in the force, namely the terms

$$-\frac{q}{2}\left(\frac{a_{\hat{\alpha}}x^2 + 2a_{\hat{\gamma}}x^{\hat{\gamma}}x_{\hat{\alpha}}}{r^3} - 3\frac{a_{\hat{\gamma}}r^{\hat{\gamma}}x^2 r_{\hat{\alpha}}}{r^5}\right).$$

If we consider the angle-average of this term, it is clear that they do not vanish, and yields

$$-\frac{q}{3r}a_{\hat{\alpha}}. \tag{2.4.7}$$

Let us take a step back for a moment and consider what we are doing. The whole goal of this procedure is to develop the equation of motion for our point particle. If we call $\langle F_\alpha \rangle$ the force constructed by taking the angle average of the full retarded solution, and any quantity $\langle Q \rangle$ to be the value of that quantity using Gralla's renormalization, we would find,

$$\langle F_\alpha \rangle = \mathtt{m}\langle a_\alpha \rangle = F^{(0)}_\alpha + F^R_\alpha - \frac{q^2 a^{(0)}_\alpha}{3r}, \tag{2.4.8}$$

where the superscript $Q^{(0)}$ is the background quantity. Since the acceleration of the particle can also be expressed in a perturbative series in the charge, we can consider

consider bringing the divergent term over to the left hand side, and, using the fact that for the background quantities their angle averaged value is the same as their actual value, we can write

$$\left(\mathtt{m} + \frac{q^2}{3r}\right) \langle a_\alpha \rangle = F^{(0)}_\alpha + F^R_\alpha. \tag{2.4.9}$$

While the above equation still has a divergent term, this term is recognizable as the *renormalized mass*, a divergent term due to the energy density of the field arising due to our assumption that the small body is in fact a point particle. As such, we have a physical justification for removing this divergent term. Therefore, we modify Gralla's angle-average prescription for geodesic motion by including also performing a mass renormalization.

Because we already argued that the angle-average is impractical for nearly all serious calculations, a natural question to ask is '*why should we even discuss the angle-average?*'

Gralla used this angle-average prescription to extend the renormalization techniques for the gravitational self-force in a Lorentz gauge to a large family of other gauges. Recently Shah and Pound [38] utilized a variant of these arguments to analyze the force and metric perturbation in a radiation gauge, one of the gauges not included in Gralla's family of regular gauges. So, even as we have tried to eschew angle-average techniques in our practical calculations, these arguments are still useful as we advance the field.

Also, by using our knowledge of the angle-average technique, we can simply pick out the elements of the retarded field which cannot contribute to the renormalized field. Instead of performing an angle average, we can generate the DW singular field by searching for all of the terms whose angle-average vanishes, and the terms that contribute to a mass-renormalization, and define the sum of these terms to be the singular field. It is this insight that we will use in Chapter 4 to analyze the renormalization techniques in electrovac.

Therefore, by using either Eq. (2.4.3) or the method of gathering the terms that vanish on angle average or contribute to the mass renormalization, we find,

$$\begin{aligned}\Phi^S &= \frac{q}{\sqrt{\hat{S}_0}} - \frac{q\hat{S}_1}{2\hat{S}_0^{3/2}} + \left\{ \frac{q}{\sqrt{\hat{S}_0}} \left[\frac{3}{8}\left(\frac{\hat{S}_1}{\hat{S}_0}\right)^2 - \frac{\hat{S}_2^{(1)}}{2\hat{S}_0} \right] \right. \\ &\quad - \frac{q}{\sqrt{\hat{S}_0}} \left[\frac{1}{6\hat{S}_0} R_{\hat{\alpha}\hat{\gamma}\hat{\beta}\hat{\delta}} u^{\hat{\gamma}} u^{\hat{\delta}} x^{\hat{\alpha}} x^{\hat{\beta}} x^{\hat{\epsilon}} x_{\hat{\epsilon}} \right] + \frac{q}{\sqrt{\hat{S}_0}} \frac{1}{12} R_{\hat{\alpha}\hat{\beta}} \left[r^{\hat{\alpha}} r^{\hat{\beta}} + u^{\hat{\alpha}} u^{\hat{\beta}} \hat{S}_0 \right] \\ &\quad \left. - \frac{1}{12} q R(z) \sqrt{\hat{S}_0} \right\}, \\ &= \Phi^{S,\mathrm{L}} + \Phi^{S,\mathrm{SL}} + \Phi^{S,\mathrm{SSL}}, \end{aligned} \tag{2.4.10}$$

Now, before moving on to consider how to use the knowledge from the local fields to generate a practical, mode-sum renormalization, we will take a slight detour to consider the singular fields for electromagnetism and gravity.

2.5 Electromagnetic and Gravitational Renormalization

In an effort to distinguish the electromagnetic vector potential from the regularization parameter A^α (from Chapter 4), we use a different font, denoting the vector potential by $\mathtt{A}^\alpha$.

We will see that, in a Lorenz gauge, each Cartesian component of the vector potential $\mathtt{A}^\alpha$ of an electric point charge and of the metric perturbation $h_{\alpha\beta}$ of a point mass has a short-distance expansion similar to that of the field of a scalar charge. We will use this similarity of form in the next Chapter to demonstrate how the properties we find for the mode-sum of the scalar self-force also extend to fields of higher spin.

We again rely on the Hadamard expansion of the Green's functions as laid out in [35].

2.5.1 Electromagnetic Self-Force

In a Lorenz gauge, the electromagnetic vector potential $\mathtt{A}^\alpha$ of a point charge e satisfies

$$\nabla^\beta \nabla_\beta \mathtt{A}^\alpha - R^\alpha{}_\beta \mathtt{A}^\beta = -4\pi j^\alpha, \quad \nabla_\alpha \mathtt{A}^\alpha = 0, \tag{2.5.1}$$

with current density

$$j^\alpha(x) = e u^\alpha(x) \int \delta^{(4)}(x, z(\tau)) d\tau. \tag{2.5.2}$$

The solution to Eq. (2.5.1) has components $\mathtt{A}^\mu$ in a global coordinate system given by

$$\mathtt{A}^{\mu}_{adv/ret}(x) = \int [G^{\mu}_{\nu'}(x,x')]_{adv/ret}\, j^{\nu'}(x')\sqrt{-g}d^4x', \qquad (2.5.3)$$

where each Green's function satisfies the equation

$$\nabla^{\gamma}\nabla_{\gamma}G^{\alpha}{}_{\hat{\beta}}(x,x') - R^{\alpha}{}_{\beta}G^{\beta}{}_{\beta'}(x,x') = -4\pi\delta^{\alpha}_{\beta'}\delta^{(4)}(x,x'). \qquad (2.5.4)$$

Unprimed and primed indices are tensor indices at x and x', respectively, and the covariant derivatives are with respect to x.

The expansion of the Green's function in the normal neighborhood C is analogous to that of the scalar field, having the form [35]

$$G^{\alpha}{}_{\beta'}(x,x') = \Theta(x,x')\left[U^{\alpha}{}_{\beta'}(x,x')\delta(\sigma) - V^{\alpha}{}_{\beta'}(x,x')\theta(-\sigma)\right], \qquad (2.5.5)$$

where the bi-tensors $U^{\alpha}{}_{\beta'}(x,x')$ and $V^{\alpha}{}_{\beta'}(x,x')$ have in RNC the local expansions

$$U^{\hat{\alpha}}{}_{\hat{\beta}}(x,x') = \delta^{\hat{\alpha}}{}_{\hat{\beta}} + \frac{1}{12}\left[2R^{\hat{\alpha}}{}_{\hat{\gamma}\hat{\beta}\hat{\delta}} + \delta^{\hat{\alpha}}_{\hat{\beta}}R_{\hat{\gamma}\hat{\delta}}\right]y^{\hat{\gamma}}y^{\hat{\delta}} + O(\epsilon^3) \qquad (2.5.6)$$

and

$$V^{\hat{\alpha}}_{\hat{\beta}} = \frac{1}{2}\left(R^{\hat{\alpha}}{}_{\hat{\beta}} - \frac{1}{6}\delta^{\hat{\alpha}}_{\hat{\beta}}R\right) + O(\epsilon). \qquad (2.5.7)$$

In these expansions, each tensor is evaluated at the point x'.

The same steps we followed for the scalar field now give for each component of $\mathtt{A}^\alpha$ essentially the same form as that of the scalar field in Eq. (2.2.7), namely

$$\mathtt{A}^{\alpha}_{adv/ret} = e\left[\frac{U^{\alpha}{}_{\beta'}u^{\beta'}}{\dot{\sigma}}\right]_{adv/ret} \mp e\lim_{h\to 0^+}\int_{\pm\infty}^{\tau_{adv/ret}\pm h} u^{\nu'}\,[G^{\alpha}{}_{\nu'}]_{adv/ret}\,d\tau. \qquad (2.5.8)$$

The force has the formal expression

$$f^{\alpha}_{EM} = -\nabla_{\beta}T^{\alpha\beta}_{EM} = F^{\alpha\beta}j_{\beta}, \qquad (2.5.9)$$

where $F_{\mu\nu} = \nabla_{\mu}\mathtt{A}_{\nu} - \nabla_{\nu}\mathtt{A}_{\mu}$, and the expression for the singular part of the force is given in terms of the singular part of the vector potential by

$$f^{S,\alpha}_{EM} = eu^{\beta}g^{\alpha\sigma}\left[\nabla_{\sigma}\mathtt{A}^{S}_{\beta} - \nabla_{\beta}\mathtt{A}^{S}_{\sigma}\right], \qquad (2.5.10)$$

where components of the metric and 4-velocity are evaluated at the position of the particle.

Now, we wish to derive the expression for the singular field. Once again, we will rely on the method we described in section 2.4.1: we will identify all of the terms whose angle average vanishes or that contribute to a mass renormalization term and define their sum to be the singular field $\mathtt{A}^S_\alpha$.

The only qualitatively new feature that the arises in the direct part of the field is the presence of the four velocity in the numerator. Consider the explicit expression for the four velocity at the retarded or advanced times:

$$u^\alpha_{ret/adv} = u^\alpha + a^\alpha(\tau_1 + \tau_2 + ...) + \frac{1}{2}\dot{a}^\alpha(\tau_1 + \tau_2 + ...)^2 + ... \tag{2.5.11}$$

By using Eqs. (2.2.18) and (2.2.19), we can rewrite $u^\alpha_{ret/adv}$ in terms of the coordinates of x as

$$\begin{aligned} u^{\hat\alpha}_{ret/adv} = u^{\hat\alpha} - a^{\hat\alpha}u_{\hat\mu}x^{\hat\mu} + \left[a^{\hat\alpha}a_{\hat\mu}u_{\hat\nu} + \frac{1}{2}\dot{a}^{\hat\alpha}(q_{\hat\mu\hat\nu} + u_{\hat\mu}u_{\hat\nu})\right]x^{\hat\mu}x^{\hat\nu} \\ \pm\left[\frac{x^{\hat\gamma}}{2}(a^{\hat\alpha}a_{\hat\gamma}(q_{\hat\mu\hat\nu} + u_{\hat\mu}u_{\hat\nu}) + 2\dot{a}^{\hat\alpha}q_{\hat\mu\hat\nu}u_{\hat\gamma}) - a^{\hat\alpha}q_{\hat\mu\hat\nu}\right]\frac{x^{\hat\mu}x^{\hat\nu}}{\sqrt{\hat{S}_0}} \end{aligned} \tag{2.5.12}$$

Therefore, we can write $u^\alpha_{ret/adv}$ in the form,

$$u^\alpha_{ret/adv} = {}_{(0)}P^\alpha + {}_{(1)}P^\alpha_\mu x^\mu + {}_{(2)}P^\alpha_{\mu\nu}x^\mu x^\nu \pm \frac{{}_{(2)}\bar{P}^\alpha_{\mu\nu}x^\mu x^\nu}{\sqrt{S_0}} \pm \frac{{}_{(3)}\bar{P}^\alpha_{\mu\nu\gamma}x^\mu x^\nu x^\gamma}{\sqrt{S_0}} + O(\epsilon^3). \tag{2.5.13}$$

Now, if we turn to $U^\alpha{}_\beta$ in Eq. (2.5.6), and we note that to leading order $y^\alpha = x^\alpha - u^\alpha\tau_1$, we can write

$$\begin{aligned} U^{\hat\alpha}{}_{\hat\beta} &= \delta^{\hat\alpha}{}_{\hat\beta} + \frac{\left(-2R^{\hat\alpha}{}_{(\hat\gamma\hat\delta)\hat\beta} + R_{\hat\gamma\hat\delta}\delta^{\hat\alpha}{}_{\hat\beta}\right)}{12}\left[x^{\hat\gamma}x^{\hat\delta} + u^{\hat\gamma}u^{\hat\delta}(S_0 + (u_{\hat\mu}x^{\hat\mu})^2)\right. \\ &\left. +2u^{\hat\gamma}x^{\hat\nu}u_{\hat\mu}x^{\hat\mu}\right] \pm \frac{u^{\hat\gamma}\left(-2R^{\hat\alpha}{}_{(\hat\gamma\hat\delta)\hat\beta} + R_{\hat\gamma\hat\delta}\delta^{\hat\alpha}{}_{\hat\beta}\right)}{6}\left[u^{\hat\delta}u_{\hat\nu}x^{\hat\nu} + x^{\hat\delta}\right]\sqrt{S_0} \end{aligned} \tag{2.5.14}$$

Notice, this can also be written in the form

$$\begin{aligned} [U^\alpha{}_\beta]_{ret/adv} &= {}_{(0)}P^\alpha{}_\beta + {}_{(1)}P^\alpha{}_{\beta\mu}x^\mu + {}_{(2)}P^\alpha{}_{\beta\mu\nu}x^\mu x^\nu \\ &\pm \frac{{}_{(2)}\bar{P}^\alpha{}_{\beta\mu\nu}x^\mu x^\nu}{\sqrt{S_0}} \pm \frac{{}_{(3)}\bar{P}^\alpha{}_{\beta\mu\nu\gamma}x^\mu x^\nu x^\gamma}{\sqrt{S_0}} + O(\epsilon^3). \end{aligned} \tag{2.5.15}$$

Using Eqs. (2.5.12) and (2.5.14), we obtain

$$
\begin{aligned}
\left[U^{\hat{\alpha}}{}_{\hat{\beta}}u^{\hat{\beta}}\right]_{ret/adv} &= u^{\hat{\alpha}} - a^{\hat{\alpha}}u_{\hat{\gamma}}x^{\hat{\gamma}} + \left[a^{\hat{\alpha}}u_{\hat{\mu}}a_{\hat{\nu}} + \frac{\dot{a}^{\hat{\alpha}}}{2}(q_{\hat{\mu}\hat{\nu}} + u_{\hat{\mu}}u_{\hat{\nu}})\right]x^{\hat{\mu}}x^{\hat{\nu}} \\
&\quad + \frac{(u^{\hat{\alpha}}R_{\hat{\gamma}\hat{\delta}} - 2R^{\hat{\alpha}}{}_{(\hat{\gamma}\hat{\delta})\hat{\beta}}u^{\hat{\beta}})}{12}\left[\delta^{\hat{\gamma}}{}_{\hat{\mu}}\delta^{\hat{\delta}}{}_{\hat{\nu}} + u^{\hat{\gamma}}u^{\hat{\delta}}(q_{\hat{\mu}\hat{\nu}} + u_{\hat{\mu}}u_{\hat{\nu}})\right. \\
&\quad \left. + 2u^{\hat{\gamma}}\delta^{\hat{\delta}}{}_{\hat{\nu}}u_{\hat{\mu}}\right]x^{\hat{\mu}}x^{\hat{\nu}} \\
&\quad \pm \frac{x^{\hat{\mu}}x^{\hat{\nu}}}{\sqrt{S_0}}\left[-a^{\hat{\alpha}}q_{\hat{\mu}\hat{\nu}} + \frac{x^{\hat{\gamma}}}{6}\left(3a^{\hat{\alpha}}(q_{\hat{\mu}\hat{\nu}} + u_{\hat{\mu}}u_{\hat{\nu}})a_{\hat{\gamma}} + 6\dot{a}^{\hat{\alpha}}u_{\hat{\gamma}}q_{\hat{\mu}\hat{\nu}}\right.\right. \\
&\quad \left.\left. + (u^{\hat{\alpha}}R_{\hat{\epsilon}\hat{\sigma}}u^{\hat{\sigma}} - 2R^{\hat{\alpha}}{}_{(\hat{\epsilon}\hat{\sigma})\hat{\beta}}u^{\hat{\beta}}u^{\hat{\sigma}})q^{\hat{\epsilon}}{}_{\hat{\gamma}}q_{\hat{\mu}\hat{\nu}}\right)\right]
\end{aligned}
\tag{2.5.16}
$$

Now, recalling Eq. (2.2.22) we can write the direct piece of the electromagnetic vector potential,

$$
\begin{aligned}
\left[\frac{U^{\hat{\alpha}}{}_{\hat{\beta}}u^{\hat{\beta}}}{\dot{\sigma}}\right]_{ret/adv} &= \frac{u^{\hat{\alpha}}}{\sqrt{S_0}}\left[1 - \frac{S_1}{2S_0} + \frac{3S_1^2}{8S_0^2} - \frac{S_2^{(1)}}{2S_0} - \frac{R_{\hat{\mu}\hat{\nu}\hat{\epsilon}\hat{\delta}}x^{\hat{\mu}}u^{\hat{\nu}}x^{\hat{\epsilon}}u^{\hat{\delta}}x^2}{6S_0}\right] \\
&\quad - \frac{a^{\hat{\alpha}}u_{\hat{\mu}}x^{\hat{\mu}}}{\sqrt{S_0}}\left(1 - \frac{S_1}{2S_0}\right) + \frac{\left[2a^{\hat{\alpha}}u_{\hat{\mu}}a_{\hat{\nu}} + \dot{a}^{\hat{\alpha}}(q_{\hat{\mu}\hat{\nu}} + u_{\hat{\mu}}u_{\hat{\nu}})\right]x^{\hat{\mu}}x^{\hat{\nu}}}{2\sqrt{S_0}} \\
&\quad + \frac{(u^{\hat{\alpha}}R_{\hat{\gamma}\hat{\delta}} - 2u^{\hat{\beta}}R^{\hat{\alpha}}{}_{(\hat{\gamma}\hat{\delta})\hat{\beta}})}{12\sqrt{S_0}}\left[\delta^{\hat{\gamma}}{}_{\hat{\mu}}\delta^{\hat{\delta}}{}_{\hat{\nu}} + u^{\hat{\gamma}}u^{\hat{\delta}}(q_{\hat{\mu}\hat{\nu}} + u_{\hat{\mu}}u_{\hat{\nu}})\right. \\
&\quad \left. + 2u^{\hat{\gamma}}\delta^{\hat{\delta}}{}_{\hat{\nu}}u_{\hat{\mu}}\right]x^{\hat{\mu}}x^{\hat{\nu}} \\
&\quad \pm \frac{x^{\hat{\mu}}x^{\hat{\nu}}}{S_0}\left[-a^{\hat{\alpha}}q_{\hat{\mu}\hat{\nu}} + \frac{x^{\gamma}}{6}\left(3a^{\alpha}(q_{\hat{\mu}\hat{\nu}} + u_{\hat{\mu}}u_{\hat{\nu}})a_{\hat{\gamma}} + 6\dot{a}^{\hat{\alpha}}u_{\hat{\gamma}}q_{\hat{\mu}\hat{\nu}}\right.\right. \\
&\quad + \left.\left.(u^{\hat{\alpha}}R_{\hat{\epsilon}\hat{\sigma}}u^{\hat{\sigma}} - 2R^{\hat{\alpha}}{}_{(\hat{\epsilon}\hat{\sigma})\hat{\beta}}u^{\hat{\sigma}}u^{\hat{\beta}})q^{\hat{\epsilon}}{}_{\hat{\gamma}}q_{\hat{\mu}\hat{\nu}}\right)\right] \pm \frac{a^{\hat{\alpha}}S_1}{2S_0} \mp u^{\hat{\alpha}}\frac{S_2^{(\pm)}}{2S_0^{3/2}},
\end{aligned}
\tag{2.5.17}
$$

where we have decomposed S_2 into two pieces, $S_2^{(1)}$, which does not change sign when switching from retarded to advanced times, and $S_2^{(\pm)}$, which does.

In the average of the retarded and advanced fields, the contribution from each term in Eq. (2.5.17) preceded by $\pm$ vanishes, so we can write the singular vector potential as,

$$\begin{aligned}\frac{1}{e}\mathbb{A}_S^{\hat\alpha} &= \frac{u^{\hat\alpha}}{\sqrt{S_0}}\left[1-\frac{S_1}{2S_0}+\frac{3S_1^2}{8S_0^2}-\frac{S_2^{(1)}}{2S_0}-\frac{R_{\hat\mu\hat\nu\hat\epsilon\hat\delta}x^{\hat\mu}u^{\hat\nu}x^{\hat\epsilon}u^{\hat\delta}x^2}{6S_0}\right]-\frac{a^{\hat\alpha}u_{\hat\mu}x^{\hat\mu}}{\sqrt{S_0}}\left(1-\frac{S_1}{2S_0}\right)\\ &+ \frac{(u^{\hat\alpha}R_{\hat\gamma\hat\delta}-2u^{\hat\beta}R^{\hat\alpha}{}_{(\hat\gamma\hat\delta)\hat\beta})}{12\sqrt{S_0}}\left[\delta^{\hat\gamma}{}_{\hat\mu}\delta^{\hat\delta}{}_{\hat\nu}+u^{\hat\gamma}u^{\hat\delta}(q_{\hat\mu\hat\nu}+u_{\hat\mu}u_{\hat\nu})+2u^{\hat\gamma}\delta^{\hat\delta}{}_{\hat\nu}u_{\hat\mu}\right]x^{\hat\mu}x^{\hat\nu}\\ &+ \frac{\left[2a^{\hat\alpha}u_{\hat\mu}a_{\hat\nu}+\dot a^{\hat\alpha}(q_{\hat\mu\hat\nu}+u_{\hat\mu}u_{\hat\nu})\right]x^{\hat\mu}x^{\hat\nu}}{2\sqrt{S_0}}+\frac{6R^{\hat\alpha}{}_{\hat\beta}u^{\hat\beta}-u^{\hat\alpha}R}{12}\sqrt{S_0}.\end{aligned} \tag{2.5.18}$$

2.5.2 Gravitational Self-Force

The test-particle limit of the trajectory of a massive particle moving in a curved spacetime is a geodesic. To consistently compute the self-force on a massive particle whose trajectory is accelerated in the test-particle limit, one must include whatever additional fields are responsible for the acceleration. Prior to the works Linz, Friedman, and Wiseman [29] and Zimmerman and Poisson [30], the study of gravitational self-force was restricted to vacuum spacetimes. In this section, we find the formal contribution from gravity to the self-force on a particle in a generic vacuum spacetime, saving the study of non-vacuum spacetimes until later (see Section 4.3).

We will write the spacetime metric as $\tilde g_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta}$, where $\tilde g_{\alpha\beta}$ is the total metric, $g_{\alpha\beta}$ is the background metric, and $h_{\alpha\beta}$ is the perturbation. We will restrict our discussion to background metrics $g_{\alpha\beta}$ that satisfy the vacuum Einstein equation. We raise and lower indices with the background metric $g_{\alpha\beta}$ and denote by ∇_α the covariant derivative operator of $g_{\alpha\beta}$.

With $\gamma_{\alpha\beta} := h_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}h$, the Lorenz gauge condition is $\nabla_\alpha\gamma^{\alpha\beta} = 0$, and the linearized Einstein equation has the form

$$\nabla_\mu\nabla^\mu\gamma^{\alpha\beta} + 2R_\gamma{}^\alpha{}_\delta{}^\beta\gamma^{\gamma\delta} = -16\pi T^{\alpha\beta}. \tag{2.5.19}$$

Here, $T^{\alpha\beta}$ is the stress energy tensor of a point particle of mass m, given by

$$T^{\alpha\beta} = m u^\alpha u^\beta \int \delta^{(4)}\left(x' - z(\tau)\right) d\tau. \tag{2.5.20}$$

As before, we write the solution to the field equation (in this case, Eq. (2.5.19)) in terms of a Green's function,

$$\gamma^{\alpha\beta} = 4\int G^{\alpha\beta}{}_{\gamma'\delta'}(x,x')T^{\gamma'\delta'}\sqrt{-g'}d^4x', \tag{2.5.21}$$

where $G^{\alpha\beta}{}_{\gamma'\delta'}(x,x')$ satisfies

$$\nabla_\mu\nabla^\mu G^{\alpha\beta}{}_{\gamma'\delta'}(x,x') + 2R_\gamma{}^\alpha{}_\delta{}^\beta G^{\gamma\delta}{}_{\gamma'\delta'}(x,x') = -4\pi g^{(\alpha}{}_{\gamma'}g^{\beta)}{}_{\delta'}\delta^4(x,x'). \tag{2.5.22}$$

As in the spin-0 and spin-1 cases, the Green's function, $G^{\alpha\beta}{}_{\gamma'\delta'}(x,x')$, has the form

$$G^{\hat{\alpha}\hat{\beta}}{}_{\hat{\gamma}\hat{\delta}}(x,x') = \Theta(x,x')\left[U^{\hat{\alpha}\hat{\beta}}{}_{\hat{\gamma}\hat{\delta}}(x,x')\delta(\sigma) - V^{\hat{\alpha}\hat{\beta}}{}_{\hat{\gamma}\hat{\delta}}(x,x')\theta(-\sigma)\right], \tag{2.5.23}$$

where the bitensors $U^{\alpha\beta}{}_{\gamma'\delta'}$ and $V^{\alpha\beta}{}_{\gamma'\delta'}$ have, in RNC about x, the expansions [35]

$$U^{\hat{\alpha}\hat{\beta}}{}_{\hat{\gamma}\hat{\delta}}(x,x') = \delta^{(\hat{\alpha}}_{\hat{\gamma}}\delta^{\hat{\beta})}_{\hat{\delta}} + \frac{1}{3}\delta^{(\hat{\alpha}}{}_{(\hat{\gamma}}R_{\hat{\delta})\hat{\sigma}}{}^{\hat{\beta})}{}_{\hat{\mu}}x^{\hat{\sigma}}x^{\hat{\mu}} + O(\epsilon^3), \tag{2.5.24a}$$

$$V^{\hat{\alpha}\hat{\beta}}{}_{\hat{\gamma}\hat{\delta}}(x,x') = R_{\hat{\gamma}}{}^{(\hat{\alpha}\hat{\beta})}{}_{\hat{\delta}} + O(\epsilon). \tag{2.5.24b}$$

When we evaluate the perturbation using Eq. (2.5.21), we find

$$\gamma^{\alpha\beta}{}_{adv/ret} = 4\mathtt{m}\left[\frac{u^{\gamma'}u^{\delta'}U^{\alpha\beta}{}_{\gamma'\delta'}}{\dot{\sigma}}\right]_{adv/ret} \mp 4\mathtt{m}\lim_{h\to 0^+}\int_{\pm\infty}^{\tau_{adv/ret}\pm h} u^{\gamma'}u^{\delta'}\left[G^{\alpha\beta}{}_{\gamma'\delta'}\right]_{adv/ret}d\tau. \tag{2.5.25}$$

Now, solving the perturbed geodesic equation allows us to write

$$f^{\alpha,S}_{GR} = -\mathtt{m}\left(g^{\alpha\delta} + u^\alpha u^\delta\right)\left(\nabla_\beta h^S_{\gamma\delta} - \frac{1}{2}\nabla_\delta h^S_{\beta\gamma}\right)u^\beta u^\gamma. \tag{2.5.26}$$

Therefore, just as for the scalar charge in Eqs. (2.2.7) and (2.2.8), and as for the electric charge in Eqs. (2.5.8) and (2.5.10), we have expressed the metric perturbation in Eq. (2.5.25) and the expression for the force in Eq. (2.5.26).

Applying the same procedure we used for $\mathtt{A}^\alpha_S$ to Eq. (2.5.24a) and solve for the retarded and advanced $\gamma_{\alpha\beta}$, we find

$$\begin{aligned}
\frac{1}{\mathtt{m}}\gamma^{ret/adv}_{\hat{\alpha}\hat{\beta}} &= \frac{4u_{\hat{\alpha}}u_{\hat{\beta}}}{\sqrt{S_0}}\left[1-\frac{S_1}{2S_0}+\frac{3S_1^2}{8S_0^2}-\frac{S_2^{(1)}}{2S_0}-\frac{R_{\hat{\mu}\hat{\nu}\hat{\epsilon}\hat{\delta}}x^{\hat{\mu}}u^{\hat{\nu}}x^{\hat{\epsilon}}u^{\hat{\delta}}x^2}{6S_0}\right] \\
&-8\frac{u_{(\hat{\beta}}a_{\hat{\alpha})}u_{\hat{\mu}}x^{\hat{\mu}}}{\sqrt{S_0}}\left(1-\frac{S_1}{2S_0}\right)+\frac{4x^{\hat{\mu}}x^{\hat{\nu}}}{\sqrt{S_0}}\Big[(a_{\hat{\alpha}}a_{\hat{\beta}}+\dot{a}_{(\hat{\alpha}}u_{\hat{\beta})})(q_{\hat{\mu}\hat{\nu}}+u_{\hat{\mu}}u_{\hat{\nu}}) \\
&+2a_{(\hat{\alpha}}u_{\hat{\beta})}a_{\hat{\mu}}u_{\hat{\nu}}-\frac{u_{(\hat{\alpha}}R_{\hat{\beta})\hat{\epsilon}\hat{\delta}\hat{\sigma}}u^{\hat{\delta}}}{3}(\delta^{\hat{\epsilon}}{}_{\hat{\mu}}\delta^{\hat{\sigma}}{}_{\hat{\nu}}+u^{\hat{\epsilon}}\delta^{\hat{\sigma}}{}_{\hat{\mu}}u_{\hat{\nu}})\Big] \\
&\pm 8u_{(\hat{\alpha}}a_{\hat{\beta})}\left(1-\frac{S_1}{2S_0}\right) \\
&\pm\frac{8x^{\hat{\mu}}x^{\hat{\nu}}x^{\hat{\delta}}}{S_0}\left[(a_{\hat{\alpha}}a_{\hat{\beta}}+\dot{a}_{(\hat{\alpha}}u_{\hat{\beta})})u_{\hat{\delta}}q_{\hat{\mu}\hat{\nu}}-a_{(\hat{\alpha}}u_{\hat{\beta})}a_{\hat{\delta}}(q_{\hat{\mu}\hat{\nu}}+u_{\hat{\mu}}u_{\hat{\nu}})\right] \\
&\mp 2\frac{u_{\hat{\alpha}}u_{\hat{\beta}}S_2^{(\pm)}}{S_0^{3/2}}\pm 4\lim_{h\to 0^+}\int_{\mp\infty}^{\tau_{ret/adv}\mp h} u^{\gamma'}u^{\delta'}\left[G_{\alpha\beta\gamma'\delta'}\right]_{adv/ret}d\tau.
\end{aligned} \tag{2.5.27}$$

Therefore, we can write the singular, trace-reversed, metric perturbation as

$$\begin{aligned}
\frac{1}{\mathtt{m}}\gamma^{S}_{\hat{\alpha}\hat{\beta}} &= \frac{4u_{\hat{\alpha}}u_{\hat{\beta}}}{\sqrt{S_0}}\left[1-\frac{S_1}{2S_0}+\frac{3S_1^2}{8S_0^2}-\frac{S_2^{(1)}}{2S_0}-\frac{R_{\hat{\mu}\hat{\nu}\hat{\epsilon}\hat{\sigma}}x^{\hat{\mu}}u^{\hat{\nu}}x^{\hat{\epsilon}}u^{\hat{\sigma}}x^2}{6S_0}\right] \\
&-8\frac{u_{(\hat{\beta}}a_{\hat{\alpha})}u_{\hat{\mu}}x^{\hat{\mu}}}{\sqrt{S_0}}\left(1-\frac{S_1}{2S_0}\right)+\frac{4x^{\hat{\mu}}x^{\hat{\nu}}}{\sqrt{S_0}}\Big[(a_{\hat{\alpha}}a_{\hat{\beta}}+\dot{a}_{(\hat{\alpha}}u_{\hat{\beta})})(q_{\hat{\mu}\hat{\nu}}+u_{\hat{\mu}}u_{\hat{\nu}}) \\
&+2a_{(\hat{\alpha}}u_{\hat{\beta})}a_{\hat{\mu}}u_{\hat{\nu}}-\frac{u_{(\hat{\alpha}}R_{\hat{\beta})\hat{\epsilon}\gamma\hat{\sigma}}u^{\gamma}}{3}(\delta^{\hat{\epsilon}}{}_{\hat{\mu}}\delta^{\hat{\sigma}}{}_{\hat{\nu}}+u^{\hat{\epsilon}}\delta^{\hat{\sigma}}{}_{\hat{\mu}}u_{\hat{\nu}})\Big] \\
&- 4u^{\hat{\mu}}u^{\hat{\nu}}R_{\hat{\mu}(\hat{\alpha}\hat{\beta})\hat{\nu}}\sqrt{S_0}.
\end{aligned} \tag{2.5.28}$$

Therefore, by subtracting the appropriate linear combination of the fields and their gradients from the retarded or advanced solutions, it is possible to develop a formal expression for the equations of motion for a particle acted on by its own self-force.

$$\begin{aligned}
f^{R,s=0}_{\alpha} &= q^2\Big[\frac{1}{3}(\dot{a}_{\hat{\alpha}}-a^2u_{\hat{\alpha}})+\frac{1}{6}R_{\hat{\beta}\hat{\gamma}}u^{\hat{\beta}}q^{\hat{\gamma}}{}_{\hat{\alpha}}-\frac{R}{12}u_{\hat{\alpha}} \\
&+\lim_{h\to 0}\int_{-\infty}^{\tau_{ret}-h}\nabla_{\hat{\alpha}}G^{ret}(z(\tau),z'(\tau')d\tau'\Big]
\end{aligned} \tag{2.5.29}$$

or, adopting the convention that $f_\alpha = q^\beta_\alpha\nabla_\beta\Phi$ so that the self-force is actually conservative, we would simply drop the Ricci scalar term and place a projection operator before the tail.

For the electromagnetic and gravitational self-forces, we find,

$$f_\alpha^{R,s=1} = e^2\left[\frac{2}{3}\left(\dot{a}_\alpha - a^2 u_\alpha\right) + \frac{1}{3}q_\alpha^\gamma R_{\gamma\beta}u^\beta + 2\delta_\alpha^{[\gamma}u^{\beta]}\lim_{h\to 0^+}\int_{-\infty}^{\tau_{ret}-h}\nabla_\beta u^{\alpha'}G_{\gamma\alpha'}d\tau'\right] \tag{2.5.30}$$

and

$$\begin{aligned} f_\alpha^{R,s=2} &= \mathtt{m}^2\left[-\frac{11}{3}(\dot{a}_\alpha - a^2 u_\alpha) + \mathtt{m}^2\left(q_\mu^\beta\left(q^{\gamma\delta} + u^\gamma u^\delta\right) - 4q_\mu^\delta u^\beta u^\gamma\right)\right. \\ &\quad \left.\times \lim_{h\to 0^+}\int_{-\infty}^{\tau_{ret}-h}\nabla_\beta G^{ret}_{\gamma\delta\alpha'\beta'}u^{\alpha'}u^{\beta'}d\tau'\right] \end{aligned} \tag{2.5.31}$$

respectively.

Chapter 3

Mode Sum Renormalization

3.1 Mode-Sum Definitions

Essentially all explicit calculations of the self-force on particles moving in Kerr or Schwarzschild geometries have used a mode-sum form of the renormalization introduced by Barack and Ori [1, 21], with early development and first applications by them, Mino, Nakano, and Sasaki and Burko [39–41]. Its subsequent development and applications by a number of researchers are reviewed by Barack [42] and Poisson et al. [35]. To regularize the mode-sum decomposition of the fields, one writes f_α^S and f_α^{ret} as sums of angular harmonics on a sphere through the particle, replacing the short-distance cutoff ρ by a cutoff ℓ_{max} in the ℓ, m harmonics, and expressing the renormalized self-force as a limit $\lim_{\ell_{\max}\to\infty}\left(\sum_{\ell=0}^{\ell_{max}} f_\alpha^{ret,\ell} - \sum_{\ell=0}^{\ell_{max}} f_\alpha^{S,\ell}\right)$ or, equivalently, as the convergent sum $\sum_{\ell=0}^{\infty}(f_\alpha^{ret,\ell} - f_\alpha^{S,\ell})$.

For a point particle with scalar charge, and, in a Lorenz gauge, for an electric charge and a point mass, $f_\alpha^{S,\ell}$ has the form

$$f_\alpha^{S,\ell,\pm} = \pm A_\alpha L + B_\alpha + \sum_{n=1}^{\infty} \frac{D_\alpha^{(2n)}}{L^{2n}}, \tag{3.1.1}$$

where the parameters A_α, B_α and the $D_\alpha^{(2n)}$ are all independent of the mode ℓ, and $L := \ell + 1/2$, and $\pm$ refers to the direction dependent expression as one approaches the particle from above or below.

A striking feature noticed by Barack and Ori [21, 22] and many other researchers is that for geodesic motion in both Schwarzschild and Kerr, the $D_\alpha^{(2n)}$ terms vanish when summed over all ℓ. meaning that an effective singular field could be defined mode by mode by writing,

$$f^{S,\ell\pm}_{eff,\alpha} = \pm A_\alpha L + B_\alpha, \tag{3.1.2}$$

where the $f^{S,\ell\pm}_{eff,\alpha}$, are the modes of the effective singular field. It is, in fact, this effective singular field (or similar ones) that is actually used in mode-sum computations. We will return to this point in Chapter 5 when we apply these principles to compute a fully renormalized self-force.

In the self-force community, it is common to write

$$D_\alpha = \sum_{\ell=0}^{\infty}\sum_{n=1}^{\infty}\frac{D^{(2n)}_\alpha}{L^{2n}}, \tag{3.1.3}$$

and say that $D_\alpha = 0$.[1]

In this Chapter, we will first discuss some properties of the spherical harmonic decomposition of smooth functions that will motivate our treatment of the singular field. Then we will introduce the mode-sum formalism and discuss some of the subtleties in how we apply it to the locally defined singular field and the specialized coordinates we will use. Then we will proceed along the same logic as the original derivation by Barack and Ori in Schwarzschild [21] to compute the A_α and B_α terms for particles moving along arbitrary trajectories in generic (smooth) background spacetimes. In doing this computation we will show that the other terms must vanish upon summing over all ℓ, meaning that the D_α term vanishes. We will continue by discussing how these results generalize to renormalizing the fields of point electric charges and point masses, giving the explicit values of the regularization parameters for the electromagnetic and gravitational self-forces. To finish the Chapter, we will first include the coordinate transformation necessary for finding the values of the regularization parameters in the original coordinate frame before discussing a refinement of the definitions for the higher order regularization parameters.

[1] In fact, the L^{-2} term is sometimes called the D term, with successively higher powers in L^{-1} taking on higher letters in the alphabet. We have adapted this notation both help distinguish between the finite term (with no superscript) and the terms falling off as finite powers of ℓ. This furthermore will ease our discussion of the higher order parameters later on when we wish to discuss terms of arbitrarily high power in L^{-1}

3.2 Mode-Sum Formalism

In mode-sum regularization one writes the retarded and singular fields as sums of angular harmonics, using the fact that the individual harmonics of the retarded field and of the expression for the self-force have finite limits on the particle's trajectory. Because the singular part of the retarded field is defined only in a normal neighborhood of the particle, its individual angular harmonics are defined only after one extends the field to a thick sphere through a position $z(0)$ of the particle. For now, we will ignore any complications introduced by the extension itself and deal with those only after conquering the rest of the mode-sum formalism.

Let (t, r, θ, ϕ) be spherical coordinates related in the usual way to a smooth Cartesian chart (t, x^1, x^2, x^3) for which the 2-spheres of constant t and r are in the domain of the chart. We denote by Φ^S any smooth extension of the singular field of Eq. (2.4.10) to a thickened 2-sphere on the $t = 0$ surface through $z(0)$ that includes a finite interval in r about the radial coordinate r_0 of $z(0)$. For Φ representing either Φ^{ret} or Φ^S, each component of the expression for the self-force along the Cartesian coordinate basis has angular harmonics $f_\alpha^{\ell m}$ given by

$$f_\alpha^{\ell m}(t, r) = q \int d\Omega \, \nabla_\alpha \Phi(t, r, \theta, \phi) \, \bar{Y}_{\ell m}(\theta, \phi), \tag{3.2.1}$$

where $\bar{Q}$ denotes the complex conjugate of the quantity Q, and $Y_{\ell,m}(\theta, \phi)$ are spherical harmonics. [2] We have seen that the renormalized self-force at $z(0)$ is given by

$$f_\alpha^R = \lim_{x \to z(0)} q \nabla_\alpha \left(\Phi^{ret} - \Phi^S \right). \tag{3.2.2}$$

[2]By using spherical harmonics, it may at first appear that we are working in a very specialized class of spacetimes, namely ones with wave equations whose angular eigenfunctions are spherical harmonics. While the examples we will draw upon in our discussion will be restricted mostly to Schwarzschild spacetime, a spacetime of this class of spacetimes, our results hold for *smooth, globally hyperbolic spacetimes.*

Regardless of the background geometry, the spherical harmonics form a complete, orthogonal basis in the angular coordinates. For example, in studies performed in Kerr spacetime, where the eigenfunctions are *spheroidal* harmonics, it is common to express the spheroidal harmonics in terms of spherical harmonics, so that the retarded field may be written in terms of spherical harmonics as well. If we instead considered a more generic spacetime geometry where the fields are difficult to write in terms of spherical harmonics, our results in this section will still be valid, although they might be more difficult to apply.

To obtain an equivalent mode-sum form of f_α^R, we first use the fact that, for $r \neq r_0$ on the thickened sphere where Φ^S is defined, Φ^{ret} and Φ^S are each smooth; second, that their angular harmonics have finite limits as $r \to r_0^\pm$ (the limits depend whether r approaches r_0 from above or below); and finally that $\nabla_\alpha \Phi^{ret} - \nabla_\alpha \Phi^S$ is continuous on the entire thickened sphere, when its value at $r = r_0$ is taken to be $\lim_{x \to z(0)}(\nabla_\alpha \Phi^{ret} - \nabla_\alpha \Phi^S)$. We then have

$$
\begin{aligned}
f_\alpha^R/q &= \lim_{r \to r_0} \nabla_\alpha \left(\Phi^{ret} - \Phi^S\right)(t=0, r, \theta_0, \phi_0) && (3.2.3)\\
&= \lim_{r \to r_0} \sum_{\ell,m} \left[\nabla_\alpha \left(\Phi^{ret} - \Phi^S\right)\right]^{\ell m}(t=0,r) Y_{\ell m}(\theta_0, \phi_0) && (3.2.4)\\
&= \sum_{\ell,m} \lim_{r \to r_0} \left[\nabla_\alpha \left(\Phi^{ret} - \Phi^S\right)\right]^{\ell m}(t=0,r) Y_{\ell m}(\theta_0, \phi_0) && (3.2.5)\\
&= \sum_{\ell,m} \left[\lim_{r \to r_0^\pm} \left(\nabla_\alpha \Phi^{ret}\right)^{\ell m}(t=0,r) - \lim_{r \to r_0^\pm} \left(\nabla_\alpha \Phi^S\right)^{\ell m}(t=0,r)\right] Y_{\ell m}(\theta_0, \phi_0), && (3.2.6)
\end{aligned}
$$

where r_0, θ_0, and ϕ_0 are the angular coordinates of the particle at time $t = 0$.

The finite range of the sum over m allows the definitions

$$
f_\alpha^{ret,\ell\pm} := q \sum_{m=-\ell}^{\ell} \lim_{r \to r_0^\pm} \nabla_\alpha \Phi^{ret,\ell m}(t=0,r) Y_{\ell m}(\theta_0, \phi_0), \qquad (3.2.7a)
$$

$$
f_\alpha^{S,\ell\pm} := q \sum_{m=-\ell}^{\ell} \lim_{r \to r_0^\pm} \nabla_\alpha \Phi^{S,\ell m}(t=0,r) Y_{\ell m}(\theta_0, \phi_0). \qquad (3.2.7b)
$$

which would allow us to write Eq. (3.2.6) as

$$
f_\alpha^R = \sum_{\ell=0}^{\infty} f_\alpha^{R,\ell} := \sum_{\ell=0}^{\infty} \left(f_\alpha^{ret,\ell\pm} - f_\alpha^{S,\ell\pm}\right). \qquad (3.2.8)
$$

In practice, when we compute the mode-sums and renormalize, we find the mode-sums of the retarded and singular fields independently. Therefore, in Eq. (3.2.8), the common practice would have us performing the difference of the sums instead of the sum of the differences. In the former case, the two sums diverge giving us a poorly defined quantity, whereas in the latter case the individual ℓ modes are finite and we can perform the subtraction.

It is possible, at this point, to perform a regulation procedure alluded to at the beginning of the Chapter and move on, but doing so would hide some useful comparisons that can be made between the mode-sum regularization techniques and the formal expressions from the previous Chapter. In order to highlight these similarities and motivate the mode-sum regularization, it is useful to briefly explore a property of spherical harmonic decompositions.

3.2.1 Large ℓ Behavior of the Harmonic Decomposition of a C^∞ Function

Claim: Let f be a C^∞ function on a domain D that includes a smoothly embedded 2-sphere S with spherical coordinates θ, ϕ. We define the spherical harmonic decomposition of f to be:

$$f = \sum_{\ell=0}^{\infty} f_\ell = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell,m} Y_{\ell,m}(\theta,\phi), \tag{3.2.9}$$

where $Y_{\ell,m}(\theta,\phi)$ are the spherical harmonics and the $f_{\ell,m}$ are given by

$$f_{\ell,m} = \int_0^\pi \sin(\theta) d\theta \int_0^{2\pi} d\phi f(\theta,\phi) \bar{Y}_{\ell,m}(\theta,\phi). \tag{3.2.10}$$

We claim that if we let k be an arbitrarily large, positive integer, then

$$\lim_{\ell\to\infty} \ell^k f_\ell = 0 \tag{3.2.11}$$

on S.

Proof: Let us define the derivative operator ${}^{(2)}\nabla^2$ to be the covariant Laplacian on S. We will now define a new function, $f^{(k)}$ by applying this operator to our function'k' times, where k is a positive integer (so $f^{(0)} = f$);

$$f^{(k)} = ({}^{(2)}\nabla^2)^k f. \tag{3.2.12}$$

Since f is C^∞ then $f^{(k)} = {}^{(2)}\nabla^{(2k)} f$ is also C^∞. Now, by extending Eqs. (3.2.9) and (3.2.10) to $f^{(k)}$, we find,

$$f^{(k)} = \sum_{\ell=0}^{\infty} f_\ell^{(k)} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell,m}^{(k)} Y_{\ell,m}(\theta,\phi), \tag{3.2.13}$$

and

$$f_{\ell,m}^{(k)} = \int_0^\pi \sin(\theta) d\theta \int_0^{2\pi} d\phi f^{(k)}(\theta,\phi) \bar{Y}_{\ell,m}(\theta,\phi). \tag{3.2.14}$$

Using the definition of $f^{(k)}$, Eq. (3.2.14) becomes

$$f^{(k)}_{\ell,m} = \int_0^{\pi} \sin(\theta)d\theta \int_0^{2\pi} d\phi(({}^{(2)}\nabla^2)^k f(\theta,\phi))\bar{Y}_{\ell,m}(\theta,\phi). \tag{3.2.15}$$

Integrating by parts 'k' times yields

$$\begin{aligned} f^{(k)}_{\ell,m} &= \int_0^{\pi} \sin(\theta)d\theta \int_0^{2\pi} d\phi f(\theta,\phi)({}^{(2)}\nabla^2)^k \bar{Y}_{\ell,m}(\theta,\phi) && (3.2.16)\\ &= \int_0^{\pi} \sin(\theta)d\theta \int_0^{2\pi} d\phi f(\theta,\phi)(\ell(\ell+1))^k \bar{Y}_{\ell,m}(\theta,\phi) && (3.2.17)\\ &= (\ell(\ell+1))^k \int_0^{\pi} \sin(\theta)d\theta \int_0^{2\pi} d\phi f(\theta,\phi)\bar{Y}_{\ell,m}(\theta,\phi) && (3.2.18)\\ &= (\ell(\ell+1))^k f_{\ell,m} && (3.2.19) \end{aligned}$$

So, since $f^{(k)}$ is C^∞ the sum over its ℓ and m modes converges, so

$$\begin{aligned} f^{(k)} &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{m=\ell} f^{(k)}_{\ell,m} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{m=\ell} \ell(\ell+1)^k f_{\ell,m} \\ &= \sum_{\ell=0}^{\infty} \ell(\ell+1)^k \sum_{m=-\ell}^{m=\ell} f_{\ell,m} = \sum_{\ell=0}^{\infty} \ell(\ell+1)^k f_\ell && (3.2.20) \end{aligned}$$

Therefore, since $f^{(k)}$ is C^∞, the sum $\sum_{\ell=0}^{\infty} \ell(\ell+1)^k f_\ell$ converges. Q.E.D.

A second property that is useful to consider, although we will not show it is that, roughly speaking, functions g on the sphere that diverge as $1/\theta^k$ near $\theta = 0$ have angular harmonics g^ℓ for which $\sum_{\ell=0}^{\ell_{max}}$ diverges as ℓ^k_{max}. [3] Therefore, the harmonic decomposition of $1/\theta^k$ will have a harmonic decomposition of the form $Const \times \ell^{k-1}$, so that when summed, the expression falls off as ℓ^k_{max}.

We will use these two insights to motivate our methods of regularizing and renormalizing the fields in the following sections.

3.3 Mode-Sum Regularization

Recall that before the brief mathematical interlude, our goal was to rewrite our expression for the renormalized force given in Eq. (3.2.2) in the form of Eq. (3.2.7b), which is to say that we wish to make the transformation,

$$\sum_{\ell=0}^{\infty} f^{ret,\ell}_\alpha - \sum_{\ell=0}^{\infty} f^{S,\ell}_\alpha \Longrightarrow \sum_{\ell=0}^{\infty}\left(f^{ret,\ell}_\alpha - f^{S,\ell}_\alpha\right) = \sum_{\ell=0}^{\infty} f^{R,\ell}_\alpha. \tag{3.3.1}$$

[3] Functions of this kind belong to Sobolev spaces H_s with $s < 0$, and the relation between the singular behavior of functions in H_s and that of their angular harmonics is described in Appendix B of [43].

In the local formulation from Chapter 2, we regularized the fields by evaluating the retarded and singular fields at a random point a distance ϵ from $z(0)$, where both fields are large but finite, and taking the limit as $\epsilon \to 0$ of the difference of these fields. Trying to do the same thing with the fields expressed in terms of spherical harmonics would be very difficult, perhaps more difficult than simply trying to perform an angle average and mass renormalization.

From the two properties listed above, though it would make sense to try to use ℓ as a regulator. In the local expansion of the fields, we argued that the singular behavior of the retarded field could be determined by the behavior of the retarded field as ϵ became very small. In this case, it should be clear that the singular behavior of the harmonic decomposition of the retarded field can be determined by examining the large ℓ behavior of the field. Or, put another way, the singular behavior of the retarded field *uniquely determines the large ℓ behavior of its angular harmonics.*

We can make sense of this by considering the DW decomposition of the retarded field, $\Phi^{ret} = \Phi^R + \Phi^S$. As we stated earlier, we will treat Φ^R as a smooth, C^∞ function of the field point, so its harmonic decomposition will fall off faster than any power of ℓ. The singular field diverges as ϵ^{-1} and so the singular force falls off as ϵ^{-2}, meaning that $f_\alpha^{S,\ell} \propto A_\alpha \ell$, where A_α is independent of ℓ.

Therefore, we introduce the regulator ℓ_{max} and write

$$f_\alpha^R = \lim_{\ell_{max}\to\infty} \left[\sum_{\ell=0}^{\ell_{max}} f_\alpha^{ret,\ell} - \sum_{\ell=0}^{\ell_{max}} f_\alpha^{S,\ell} \right] \tag{3.3.2a}$$

$$= \lim_{\ell_{max}\to\infty} \sum_{\ell=0}^{\ell_{max}} \left[f_\alpha^{ret,\ell} - f_\alpha^{S,\ell} \right] \tag{3.3.2b}$$

$$= \sum_{\ell=0}^{\infty} \left[f_\alpha^{ret,\ell} - f_\alpha^{S,\ell} \right], \tag{3.3.2c}$$

which implies that $f_\alpha^{R,\ell} = f_\alpha^{ret,\ell} - f_\alpha^{S,\ell}$. Now that we have considered the regularization, we can renormalize the regularized field.

3.4 Mode-Sum Renormalization

Now that we have demonstrated how to actually perform the subtraction between the two divergent quantities $\nabla_\alpha \Phi^{ret} - \nabla_\alpha \Phi^S$, all that remains is to actually take this difference. We will now proceed by assuming that the retarded field is known, either through some numerical process or through the analytic process given in chapters 5 and 6, and focus purely on the mode-sum decomposition of the singular field.

For our purposes, we do not even need the full functional decomposition of the singular field, we just need to know the value of $\nabla_\alpha \Phi^{S,\ell}_\alpha$ evaluated at the position of the particle. Recalling the arguments about the ℓ dependences of the spherical harmonic decomposition of a field evaluated at a point where the field is divergent, we anticipate that the singular field will have the form

$$f^{S,\ell}_\alpha = A_\alpha L + B_\alpha + C_\alpha L^{-1} + O(L^{-2}), \tag{3.4.1}$$

where A_α, B_α, and C_α constants independent of ℓ. The leading term, $A_\alpha L$, arise sfrom the $1/\epsilon^2$ (Coulomb) behavior of f^{ret}_α. The B_α term arises from the $1/\epsilon$ behavior of the mass-renormalization terms and corrections to the coulomb term. A term C_α/L would yield a logarithmic divergence in the sum

$$\sum_{\ell=0}^{\ell_{max}} C_\alpha/L = C_\alpha \log \ell_{max} + O(\ell_{max}^{-1});$$

because this would correspond to a (nonexistent) $\log \epsilon$ term in the short-distance expansion of f^{ret}_α, it cannot be present. The argument can be made precise:[4] After subtracting the leading and subleading terms from the singular field, the remainder is defined and uniformly bounded everywhere on the sphere except at a point (the position of the particle), where it is direction-dependent. Its angular transform is therefore convergent, implying that no term of the form $1/L$ can be present. Our calculation in Sec. 3.4.1 below explicitly verifies that $C_\alpha = 0$.

Finally, terms of order ϵ^0 in f^S_α (terms of order L^{-2} or higher, including terms falling off faster than any power of L) could in principle contribute to D_α,

$$D_\alpha = \sum_{\ell=0}^{\infty} (f^{S,\ell}_\alpha - A_\alpha L + B_\alpha). \tag{3.4.2}$$

[4]This was pointed out to us by Sam Gralla

Following [21], we refer to $A_\alpha, B_\alpha, C_\alpha$ and D_α as 'regularization parameters'.[5]

For a scalar charge undergoing geodesic motion in Schwarzschild spacetimes Barack and Ori [21] demonstrated that the D_α term vanishes. This means that we can truncate the expansion of the singular field in powers of ℓ^{-1} and write an effective singular field as

$$f_\alpha^{S,\ell\pm} = \pm A_\alpha L + B_\alpha. \tag{3.4.3}$$

where $L \equiv \ell + 1/2$, and A_α and B_α are constants independent of ℓ. Other work by Barack and Ori [44] and Warburton and Barack [45], [46], demonstrated that this form holds in Kerr spacetime as well. It has also been demonstrated that the electromagnetic and gravitational regularization parameters also have this convenient form (in a Lorentz gauge for gravity) [22], [42].

In this Chapter, we will demonstrate the main result from Linz, Friedman, and Wiseman [24] and demonstrate that we can extend these results to arbitrarily accelerated trajectories in smooth, globally hyperbolic spacetimes.

Because Eqs. (3.2.7) involve sums over all m, the values of $f_\alpha^{ret,\ell\pm}$ and $f_\alpha^{S,\ell\pm}$ are invariant under a rotation of the (θ, ϕ) coordinates. To evaluate them, it is convenient to choose rotated coordinates (that we again denote by θ, ϕ) for which the particle is on the coordinate axis, $\theta = 0$ at $z(0)$ (see Fig 3). Using $Y_{\ell m}(\theta = 0, \phi) = 0 \ \forall \ m \neq 0$ and Eqs. (3.2.1) and (3.2.7b), we can write

$$f_\alpha^{S,\ell\pm} \equiv \left[\nabla_\alpha \Phi^S\right]^\ell = \lim_{r \to r_0^\pm} \frac{L}{2\pi} \int d\Omega P_\ell(\cos(\theta)) \nabla_\alpha \Phi^S. \tag{3.4.4}$$

Therefore, to calculate the regularization parameters, we will use Eq. (3.4.4), with Φ^S given by Eq. (2.4.10). We will then group the terms as ones that are linear in L, independent of L, inversely proportional to L and proportional to L^{-2n}, and identify these with the A_α, B_α, C_α and $D_\alpha^{(2n)}$ terms respectively. We then perform the sum over all ℓ of the $D_\alpha^{(2n)}$ terms and this will give us the D_α term.

[5] In [24] we wrote this as Δ_α in an effort to dispel the growing confusion between an overall leftover term and the coefficient of the L^{-2} term. As the old terminology has stuck, we will revert to this definition to be in keeping with the self-force community.

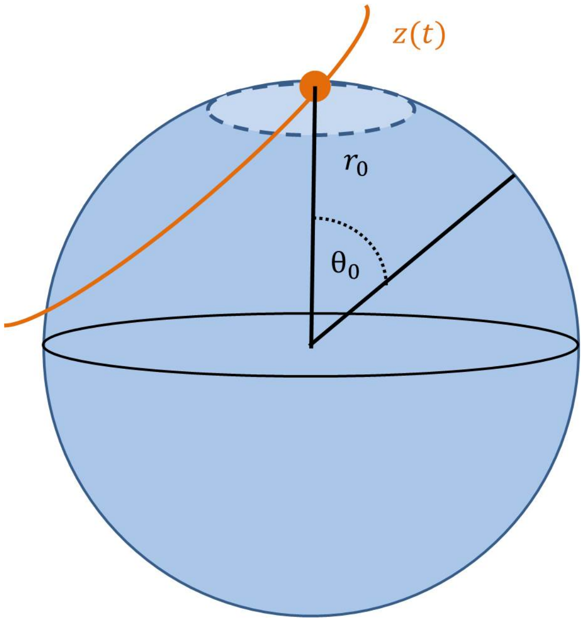

Figure 3: The particle is shown at time $\tau = t = 0$, at a coordinate distance r_0 from the origin. We rotate our coordinates by an angle θ_0 so that the particle is placed at the north pole. The small region bordered by the dashed line represents the region in which the singular field is well defined–the normal neighborhood of the particle.

From Eq. (2.4.10), the singular field's leading order term is $O(\epsilon^{-1})$, and the leading-order term in its derivative is $O(\epsilon^{-2})$. Recalling Eq. (3.4.4), we write

$$f_\alpha^{S,\ell} = f_\alpha^{\mathrm{L},\ell} + f_\alpha^{\mathrm{SL},\ell} + f_\alpha^{\mathrm{SSL},\ell}, \tag{3.4.5}$$

where $f_\alpha^{\mathrm{L},\ell}$, $f_\alpha^{\mathrm{SL},\ell}$, and $f_\alpha^{\mathrm{SSL},\ell}$ denote respectively the contributions to f_α^S at leading, sub-leading, and sub-subleading order. From Eq. (2.4.10), they are given by the following expressions, evaluated on the $t = 0$ surface:

$$f_\alpha^{\mathrm{L},\ell} = \lim_{r\to r_0^\pm} q \int d\Omega P_\ell(\cos(\theta)) \nabla_\alpha \Phi^{\mathrm{L}}, \tag{3.4.6a}$$

$$f_\alpha^{\mathrm{SL},\ell} = \lim_{r\to r_0^\pm} q \int d\Omega P_\ell(\cos(\theta)) \nabla_\alpha \Phi^{\mathrm{SL}}, \tag{3.4.6b}$$

$$f_\alpha^{\mathrm{SSL},\ell} = \lim_{r\to r_0^\pm} q \int d\Omega P_\ell(\cos(\theta)) \nabla_\alpha \Phi^{\mathrm{SSL}}. \tag{3.4.6c}$$

In the remainder of this section, we use Eq. (3.4.6), with Φ^{L}, Φ^{SL}, and Φ^{SSL} given by Eq. (2.4.10), to show that the large ℓ behavior of f_α^S given in Eq. (3.4.3) follows from the general character of the short-distance form of Φ^S, given in Eqs. (3.4.7) below. We then find the explicit forms of A_α and B_α. Denoting by $P^{(k)}(x^\mu)$ a homogeneous polynomial of degree k in the coordinates x^μ, we write the leading, subleading, and sub-subleading terms of Φ^S in the form

$$\Phi^{\rm L} = \frac{C}{\hat{S}_0^{1/2}} \quad (3.4.7a)$$

$$\Phi^{\rm SL} = \frac{P^{(3)}(x^\mu)}{\hat{S}_0^{3/2}} \quad (3.4.7b)$$

$$\Phi^{\rm SSL} = \frac{P^{(6)}(x^\mu)}{\hat{S}_0^{5/2}}. \quad (3.4.7c)$$

For $\Phi^{\rm L}$ and $\Phi^{\rm SL}$, this form is explicit in Eq. (2.4.10); for $\Phi^{\rm SSL}$, terms are grouped with the common denominator $S_0^{5/2}$.

That the mode-sum expression (3.4.3) holds for electromagnetic and gravitational perturbations will again follow from the fact that each component of the corresponding singular fields (the singular parts of the perturbed vector potential and metric) satisfies Eq. (3.4.7).

There is a subtlety that we have been ignoring here. In figure 3, we depicted the normal neighborhood of the point $z(0)$ by a dashed line, and this region will typically not extend to the entire surface of the sphere. Unfortunately, the singular field is only properly defined in this region, so we must seek a way to extend the field from this region to the entire sphere.

Because the mode-sum involves spherical harmonics associated with a specified coordinate system (t, r, θ, ϕ), we begin by rewriting the short-distance expansion Eq. (2.4.10) as an expansion in terms of the coordinate distances to the particle. To do so, we define Cartesian coordinates x^μ (termed "locally Cartesian angular coordinates" in [21]) associated with these coordinate differences by

$$x^0 = t, \quad x^1 = x = \rho(\theta)\cos\phi \quad x^2 = y = \rho(\theta)\sin\phi, \quad x^3 = r - r_0, \quad (3.4.8)$$

where $\rho(\theta) = 2\sin(\theta/2)$. In choosing these coordinates – in particular, choosing $\rho(\theta)$ instead of $\sin\theta$ – and in subsequently discarding terms of order ϵ^2, we need to check that different choices give the same angular harmonic series up to convergent terms whose sum vanishes at the particle. We can see that this is the case, because two choices of

$\rho(\theta)$ that differ by terms of order θ^3 and for which the corresponding values of $\nabla\rho$ differ by $O(\theta^2)$ give expansions of each component $\nabla_\alpha\Phi^S$ that differ by a continuous function that is $O(\epsilon)$. The difference in the angular harmonic series of each component $\nabla_\alpha\Phi^S$ is therefore a series that converges to zero at the particle. The values of the regularization parameters A_α and B_α, regarded as vectors, depend on the original coordinate system (t, r, θ, ϕ), but not on the locally Cartesian coordinates we use to evaluate them. Their components, of course, depend on the choice of basis.

Another way of interpreting this is as follows: we are not choosing the Detweiler-Whiting singular field here, but we are choosing a different field Ψ^S such that $\nabla_\alpha\nabla^\alpha\Psi^S = \nabla_\alpha\nabla^\alpha\Phi^{ret} = -4\pi\delta(x - z(\tau))$ and $\nabla_\alpha(\Phi^S - \Psi^S) = 0$, meaning that this field satisfies the two conditions required for a singular field.

In the language of mode-sum renormalization, if this field differs from the singular field by a C^∞ then the large ℓ expansions of the fields will remain unchanged. This means that A_α, B_α, and C_α will remain unchanged. In fact, for all finite n, this means that the $D_\alpha^{(2n)}$ terms will be unchanged also. On the other hand, it is possible that by choosing a different extension, we could introduce a term that falls off faster than any power of ℓ that *does not vanish when summed over all ℓ*. In order to ensure that this is not the case, we choose $\rho(\theta)$ such that it only differs from $\sin(\theta)$ at order θ^3, so that we know that as we approach the particle, this term will indeed vanish.

The coordinates x^μ are related to RNCs $x^{\hat{\alpha}}$ by

$$x^{\hat{\alpha}} = \partial_\mu x^{\hat{\alpha}} x^\mu + \frac{1}{2}\partial_\mu x^{\hat{\alpha}}\Gamma^\mu{}_{\epsilon\nu}x^\epsilon x^\nu + \frac{1}{6}\partial_\mu x^{\hat{\alpha}}\left(\Gamma^\mu{}_{\nu\gamma}\Gamma^\gamma{}_{\epsilon\lambda} + \partial_\lambda\Gamma^\mu{}_{\nu\epsilon}\right)x^\epsilon x^\nu x^\lambda + \ldots \tag{3.4.9}$$

When we use this relation to replace the RNCs by the coordinates x^μ, the expansion Eq. (2.4.10) retains the same form, with $\hat{S}_0$, $\hat{S}_1$, and $\hat{S}_2$ replaced by quantities S_0, S_1, and S_2, where

$$S_0 := q_{\mu\nu}x^\mu x^\nu, \tag{3.4.10}$$

$$S_1 := \left(a_\lambda g_{\mu\nu} + \frac{1}{2}g_{\mu\nu,\lambda} + u_\epsilon u_\lambda \Gamma^\epsilon{}_{\mu\nu}\right)x^\mu x^\nu x^\lambda =: 2\zeta_{\mu\nu\lambda}x^\mu x^\nu x^\lambda, \tag{3.4.11}$$

with all quantities in parentheses evaluated at $z(0)$. We will not use the explicit expression for S_2 and do not give it here because of its length; we need only the fact that it is a homogeneous polynomial of degree 4 in the coordinates x^μ.

Our treatment of Eqs. (3.4.6b) in section 3.4.2 and (3.4.6c) in section 3.4.1 differs from that of Eq. (3.4.6a) in section 3.4.3. In the former cases, we are allowed to take the limit inside the integral, which simplifies the calculation. In the latter case we cannot do this. The fact that the limit and integral commute follows from the fact that, after one writes $d\Omega = d\theta d\phi \sin\theta$, the integrands in Eqs. (3.4.6b) and (3.4.6c) are bounded functions of θ and ϕ and are defined everywhere except at $\theta = 0$.[6] We examine these subleading and sub-subleading terms before evaluating the leading term.

Throughout this section, we have been following the methods of Barack and Ori [21] exactly. At this point they used properties of the Schwarschild geometry, and we rephrase the argument in a way that holds for a general background spacetime.

3.4.1 The Sub-Sub-Leading term

The sub-subleading term in the self-force is the easiest to evaluate, and we will see that it vanishes. A function $\Phi^{\rm SSL}$ of the form (3.4.7c) has gradient of the form

$$\nabla_\alpha \Phi^{\rm SSL} = \frac{P_\alpha^{(7)}(x^\mu)}{S_0^{7/2}}, \qquad (3.4.12)$$

where each component $P_\alpha^{(7)}$ is a homogeneous polynomial of degree 7. Because only polynomials in the three coordinates x^i, $i = 1, \ldots, 3$ survive when $f_\alpha^{\rm SSL,\ell}$ is evaluated on the $t = 0$ surface, we have

$$f_\alpha^{\rm SSL,\ell} = \lim_{r\to r_0} \frac{q^2 L}{2\pi} \int d\Omega P_\ell(\cos(\theta)) \frac{P_\alpha^{(7)}(x^i)}{S_0^{7/2}}. \qquad (3.4.13)$$

That a function of the form $P^{(k)}(x^i)/S_0^{k/2}$ is bounded follows immediately from the definition (3.4.10) of S_0 and the fact that the spatial part q_{ij} of $q_{\mu\nu}$ is positive definite. As noted above, we can then interchange the order of the limit and integration. To see that the integral over the sphere at $r = r_0$ vanishes, we use the fact that $P^{(7)}$ is odd under $I : x^\mu \to -x^\mu$, while S_0 is even (see the specific discussion in next section, after Eq. (3.4.17)). From Eq. (3.4.8) the restriction of I to the $t = 0$, $r = r_0$ sphere is the map

[6] The result is an immediate consequence of Lebesgue's dominated convergence theorem (see, for example, [47], p. 191): Let $\{F_n\}$ be a sequence of integrable functions that converges almost everywhere to F. If $|F| < G$, for some integrable function G, then F is integrable and $\int F d\mu = \lim_{n\to\infty} \int F_n d\mu$. For functions of the type we consider here, a proof can also be found in [21].

$\phi \to \phi + \pi$, implying that the sphere itself and the measure $d\Omega$ are invariant under I. Because the integrand is odd under I and $d\Omega$ is invariant, the integral vanishes.

3.4.2 The Subleading Term

The subleading term of Eq. (3.4.6b),

$$f_\alpha^{\mathrm{SL},\ell} = \lim_{r\to r_0^\pm} \frac{q^2 L}{2\pi} \int d\Omega P_\ell(\cos(\theta)) \nabla_\alpha \left(-\frac{S_1}{2S_0^{3/2}} \right), \tag{3.4.14}$$

is more singular than the sub-subleading term by an additional power of $S_0^{1/2}$ in its denominator. It has the form

$$f_\alpha^{\mathrm{SL},\ell} = \lim_{r\to r_0^\pm} \frac{q^2 L}{2\pi} \int d\theta d\phi \sin\theta P_\ell(\cos\theta) \frac{P_\alpha^{(2n)}(x^i)}{S_0^{n+1/2}}, \tag{3.4.15}$$

To compute the explicit form of $f_\alpha^{SL,\ell}$ and to see that $\sin\theta \frac{P_\alpha^{(2n)}(x^i)}{S_0^{n+1/2}}$ is bounded, we begin by noting that, restricted to the $r = r_0, t = 0$ sphere, $P_\alpha^{(2n)}$ and S_0 are given by

$$P_\alpha^{(2n)}(x^i)\big|_{r=r_0} = \rho(\theta)^{2n} \left(\sum_{m=0}^{2n} a_{\alpha,m} \sin^m\phi \cos^{2n-m}\phi \right), \tag{3.4.16}$$

where $a_{\alpha,m}$ is a constant; and

$$\tilde{S}_0 := S_0|_{r=r_0} = \rho(\theta)^2 \left(q_{xx}\cos(\phi)^2 + q_{yy}\sin(\phi)^2 \right), \tag{3.4.17}$$

where we have used the fact that, with our rotated θ, ϕ coordinates, $q_{xy} = 0$. In effect, this is exactly what Barack and Ori [21] do for Schwarzschild, choosing their coordinates such that $u_y = 0$, and then relying on the diagonal form of the metric to make $q_{xy} = 0$. Then, because the eigenvalues of q_{IJ}, $I, J = 1\ldots2$, are positive definite, S_0 can be written as

$$\tilde{S}_0 = \rho(\theta)^2 q_{yy} \left(1 + \beta^2\cos^2\phi \right), \tag{3.4.18}$$

where

$$\beta^2 := \frac{q_{xx} - q_{yy}}{q_{yy}}. \tag{3.4.19}$$

From Eqs. (3.4.16) and (3.4.18), it follows that $S_0^{n+1/2}$ has one more power of $\rho(\theta)$ than $P_\alpha^{(2n)}$ and hence that the integrand, $\sin\theta P_\ell(\cos\theta) P_\alpha^{(2n)} S_0^{-(n+1/2)}$, is bounded.

We can therefore again bring the limit inside the integral in Eq. (3.4.15). Substituting the expressions (3.4.16) and (3.4.18) for $P_\alpha^{(2n)}$ and $\tilde{S}_0$ in Eq. (3.4.15), we have

$$f_\alpha^{\mathrm{SL},\ell} = \frac{q^2 L}{2\pi {q_{yy}}^{n-1/2}} \int_0^\pi d\theta \sin\theta \frac{P_\ell(\cos(\theta))}{\rho(\theta)} \sum_{m=0}^{2n} \int_0^{2\pi} \frac{(a_{\alpha,m} \sin^m \phi \cos^{2n-1-m} \phi)}{(1+\beta^2\cos^2\phi)^{(2n-1)/2}} d\phi. \quad (3.4.20)$$

The integral over θ has the value

$$\int_0^\pi d\theta \sin\theta \frac{P_\ell(\cos(\theta))}{\rho(\theta)} = \int_0^\pi d\theta \sin\theta \frac{P_\ell(\cos(\theta))}{\sqrt{2-2\cos(\theta)}} = \frac{1}{L}, \quad (3.4.21)$$

implying $f_\alpha^{\mathrm{SL},\ell}$ is independent of ℓ:

$$f_\alpha^{\mathrm{SL},\ell} = B_\alpha. \quad (3.4.22)$$

The integration over ϕ involves the complete elliptic integrals

$$E(w) = \int_0^{\pi/2} (1 - w\sin^2\phi)^{1/2} d\phi, \qquad K(w) = \int_0^{\pi/2} (1 - w\sin^2\phi)^{-1/2} d\phi, \quad (3.4.23)$$

where

$$w := \frac{\beta^2}{1+\beta^2}. \quad (3.4.24)$$

After a straightforward computation, we find

$$B_\alpha = \frac{2q^2}{3\pi(1+\beta^2)^{3/2}\beta^4 q_{yy}^{5/2}} \left(B_\alpha^{(E)} E(w) + B_\alpha^{(K)} K(w) \right), \quad (3.4.25)$$

where

$$B_\alpha^{(E)} = (1+\beta^2)(2+\beta^2)\Lambda_{\alpha XXYY} - 2\Big[(1+2\beta^2)\Lambda_{\alpha xxxx} + (1+\beta^2)^2(1-\beta^2)\Lambda_{\alpha yyyy}\Big], \quad (3.4.26a)$$

$$B_\alpha^{(K)} = (2+3\beta^2)\Lambda_{\alpha xxxx} + (1+\beta^2)\left[(2-\beta^2)\Lambda_{\alpha yyyy} - 2\Lambda_{\alpha XXYY}\right], \quad (3.4.26b)$$

with the quantities $\Lambda_{\alpha\beta\gamma\delta\epsilon}$ given in terms of $\zeta_{\beta\gamma\delta}$ of Eq. (3.4.11) by

$$\Lambda_{\alpha\beta\gamma\delta\epsilon} := 3\zeta_{(\alpha\beta\gamma)} q_{\delta\epsilon} - 3\zeta_{\beta\gamma\delta} q_{\alpha\epsilon}, \quad (3.4.27)$$

and we define the $\Lambda_{\alpha XXYY}$ as follows;

$$\Lambda_{\alpha XXYY} = \Lambda_{\alpha xxyy} + \Lambda_{\alpha xyxy} + \Lambda_{\alpha xyyx} + x \leftrightarrow y. \quad (3.4.28)$$

In summary, we have shown that the angular harmonic decomposition of the subleading term has only a B term, a term independent of ℓ, whose explicit form is given by Eqs. (3.4.25)-(3.4.27).

These parameters agree with those of Barack and Ori for Schwarzschild [21], and also with Warburton and Barack [45] and [46] in Kerr. (In particular note the equivalence of our Eq. (3.4.27) with Eqs. (B5), (B6) and (B7) of [45]).

3.4.3 Leading Term

Finally, we turn to the leading term $f_\alpha^{\mathrm{L},\ell}$. From Eq. (3.4.6a) and the relation $\nabla_\alpha S_0 = 2q_{\alpha\beta}x^\beta$, we have

$$f_{\alpha\pm}^{\mathrm{L},\ell} = -\frac{L}{2\pi}q^2 q_{\alpha\beta}\tilde{F}_\pm^{\beta\ell}, \tag{3.4.29}$$

where

$$\tilde{F}_\pm^{\beta\ell} = \lim_{r\to r_0^\pm}\int d\Omega P_\ell(\cos(\theta))\frac{x^\beta}{S_0^{3/2}}. \tag{3.4.30}$$

Because we are working on a $t=0$ surface, we have $\tilde{F}_\pm^{0\ell} = 0$. To evaluate $\tilde{F}_\pm^{i\ell}$, we follow Barack and Ori [21], dividing the $r =$ constant sphere that constitutes the domain of integration into two parts: the coordinate square $\mathcal{S}_\epsilon$ for which $|x| < \epsilon$ and $|y| < \epsilon$ (some $\epsilon < \pi/2$); and the rest of the sphere, $S^2\backslash\mathcal{S}_\epsilon$. The domains are chosen to be symmetric under a rotation by π about $\theta = 0$.

On $S^2\backslash\mathcal{S}_\epsilon$, the integrand is smooth, and we can bring the limit inside the integral, writing

$$\lim_{r\to r_0^\pm}\int_{S^2\backslash\mathcal{S}_\epsilon} d\Omega P_\ell(\cos(\theta))\frac{x^i}{S_0^{3/2}} = \int_{S^2\backslash\mathcal{S}_\epsilon} d\Omega P_\ell(\cos(\theta))\frac{x^i}{\tilde{S}_0^{3/2}}.$$

We immediately see that the contribution to the radial component $\tilde{F}_\pm^{1\ell}$ vanishes. The remaining x and y components of the integral vanish because the domain of integration and the function $\tilde{S}_0$ are invariant under a rotation by π about $\theta = 0$, while x and y change sign.

The only contribution to $\tilde{F}_\pm^{\beta\ell}$ is then from the integral over $\mathcal{S}_\epsilon$. Because ϵ is arbitrary, the value of the integral is independent of ϵ, determined only by the singular behavior of the integrand at $\theta = 0$. To evaluate the integral, we change integration variables from

(θ, ϕ) to (x, y). From Eq. (3.4.8), the Jacobian of the transformation is

$$\frac{\partial(\theta, \phi)}{\partial(x, y)} = \sin\theta, \tag{3.4.31}$$

and we have

$$\tilde{F}_{\pm}^{i\ell} = \lim_{r\to r_0^{\pm}} \int_{\mathcal{S}_\epsilon} dxdy P_\ell(\cos\theta) \frac{x^i}{S_0^{3/2}} = \lim_{r\to r_0^{\pm}} \int_{-\epsilon}^{\epsilon} dx \int_{-\epsilon}^{\epsilon} dy P_\ell(\cos\theta) \frac{x^i}{S_0^{3/2}}. \tag{3.4.32}$$

Because $P_\ell(\cos\theta)$ differs from its value at $\theta = 0$ only at $O(\theta^2)$, replacing P_ℓ by 1 does not alter the leading singular behavior of the integrand and should therefore not change the value of the integral. To verify this, we write

$$P_\ell(\cos\theta) = 1 + h(\theta)\sin^2\theta, \tag{3.4.33}$$

where h is smooth on $\mathcal{S}_\epsilon$. We then have

$$\tilde{F}_{\pm}^{i\ell} = \lim_{r\to r_0^{\pm}} \int_{\mathcal{S}_\epsilon} dxdy \frac{x^i}{S_0^{3/2}} + \int_{\mathcal{S}_\epsilon} dxdy \lim_{r\to r_0^{\pm}} \left(h\sin^2\theta \frac{x^i}{S_0^{3/2}} \right) \equiv (\lim_{r\to r_0^{\pm}} I_1^i) + I_2^i,$$

where we have used the fact that the function $h\sin^2\theta \; x^i/S_0^{3/2}$ is bounded to bring the limit inside the second integral, I_2^i. Then I_2^i has the form

$$I_2^i = \int_{\mathcal{S}_\epsilon} dxdy \left(h\sin^2\theta \frac{x^i}{\tilde{S}_0^{3/2}} \right). \tag{3.4.34}$$

Again the vanishing of I_2^r is immediate, and the symmetry argument we have now used twice implies that the remaining components also vanish: That is, from the invariance of $\mathcal{S}_\epsilon$ and $h\sin^2\theta/\tilde{S}_0^{3/2}$ under a π rotation, together with the fact that x and y change sign, we have $I_2^x = I_2^y = 0$.

We are now left with

$$\tilde{F}_{\pm}^{i\ell} = \lim_{r\to r_0^{\pm}} \int_{\mathcal{S}_\epsilon} \frac{x^i}{S_0^{3/2}} dxdy. \tag{3.4.35}$$

We can already see that this integral is independent of L, because P_ℓ has been replaced by 1. It immediately follows from Eq. (3.4.29) that $f_\alpha^{L,\ell}$ is proportional to L, and *we have thus established our central claim, that the singular part of the self-force has the form given in Eq. (3.1.2).*

Finally, we evaluate $\tilde{F}_{\pm}^{i\ell}$ to find the explicit form of A_α. We begin by showing that the x- and y-components can be expressed in terms of the third spatial component $\tilde{F}_{\pm}^{r\ell}$.

From the definition (3.4.10) of S_0, we have

$$\partial_x \frac{1}{S_0^{1/2}} = -\frac{q_{xx}x + q_{xr}(r-r_0)}{S_0^{3/2}}, \tag{3.4.36}$$

and the x-component of Eq. (3.4.35) takes the form

$$\tilde{F}_{\pm}^{x\ell} = -\frac{1}{q_{xx}} \lim_{r\to r_0^{\pm}} \int_{\mathcal{S}_\epsilon} \left[\partial_x \frac{1}{S_0^{1/2}} + \frac{q_{xr}}{S_0^{3/2}}(r-r_0) \right] dxdy. \tag{3.4.37}$$

Using $\displaystyle\int_{-\epsilon}^{\epsilon} dx\ \partial_x S_0^{-1/2} = 0$, we have

$$\tilde{F}_{\pm}^{x\ell} = -\frac{q_{xr}}{q_{xx}} \lim_{r\to r_0^{\pm}} \int_{\mathcal{S}_\epsilon} \frac{r-r_0}{S_0^{3/2}} dxdy = -\frac{q_{xr}}{q_{xx}} \tilde{F}_{\pm}^{r\ell}, \tag{3.4.38}$$

as claimed. Similarly,

$$\tilde{F}_{\pm}^{x\ell} = -\frac{q_{yr}}{q_{yy}} \tilde{F}_{\pm}^{r\ell}. \tag{3.4.39}$$

To evaluate $\tilde{F}_{\pm}^{r\ell}$, we introduce as integration variables

$$X = \frac{x}{r-r_0}, \quad Y = \frac{y}{r-r_0}. \tag{3.4.40}$$

With $e : \epsilon/(r-r_0)$, we have

$$\begin{aligned} \tilde{F}_{\pm}^{r\ell} &= \lim_{e\to\infty} \int_{-e}^{e} dX \int_{-e}^{e} dY [q_{xx}X^2 + 2q_{xr}X + q_{yy}Y^2 + 2q_{yr}Y + q_{rr}]^{-3/2} \\ &= \pm 2\pi (q_{xx}q_{yy}q_{rr} - q_{yy}q_{xr}^2 - q_{xx}q_{yr}^2)^{-1/2}. \end{aligned} \tag{3.4.41}$$

Finally, using $f_{\alpha\pm}^{L,\ell} = A_\alpha L$, together with Eqs. (3.4.29), (3.4.38), (3.4.39) and (3.4.41), we obtain

$$A_{\alpha\pm} = \mp q^2 \, \frac{q_{\alpha r} - q_{\alpha x}q_{xr}/q_{xx} - q_{\alpha y}q_{yr}/q_{yy}}{(q_{xx}q_{yy}q_{rr} - q_{yy}q_{xr}^2 - q_{xx}q_{yr}^2)^{1/2}}. \tag{3.4.42}$$

It is worth noting that this agrees with the form given in [21] and also has the same property that $u_\alpha A^\alpha = 0$.

Thus, as claimed, the regularization parameters for the self force on a point scalar charge moving along an arbitrary trajectory through a generic spacetime are given by $A_\alpha L + B_\alpha$, with the terms for a logarithmic divergence ($C_\alpha L^{-1}$) and a finite remainder (D_α) both vanishing. We have given the explicit forms of the regularization parameters in the 'locally Cartesian angular coordinates,' in Eqs. (3.4.42) and (3.4.25). Their values for the original coordinate system are given later in this Chapter.

It is important to note that we have recovered the regularization parameters for $f^{S,\ell}_{\alpha}$, whose values are not (necessarily) trivially related to those for $f^{S,\ell,\alpha}$. For now we will just claim that the parameters for the raised indices, the regularization parameters have the form, $A^{\alpha}L + B^{\alpha}$, and postpone the proof to the end of next section, where we can discuss it in the context of extending the four velocity away from the world-line.

3.5 Regularization Parameters for Electromagnetism and Gravity

Here we write the explicit regularization parameters for the self-force on a point electric charge and a point mass (computed in a Lorenz gauge). We directly parallel the approach taken for the scalar charge.

3.5.1 Electromagnetic Regularization Parameters

Until the final equation of this section, we set the charge e to 1.

We begin by writing Eq. (2.5.18), but we keep only the leading and sub-leading terms

$$\mathrm{A}^{S}_{\hat{\alpha}} = \frac{u_{\hat{\alpha}}}{\sqrt{\hat{S}_0}} - \frac{\left[u_{\hat{\alpha}}\zeta_{\hat{\gamma}\hat{\delta}\hat{\epsilon}} + a_{\hat{\alpha}}u_{\hat{\gamma}}\left(\eta_{\hat{\epsilon}\hat{\delta}} + u_{\hat{\epsilon}}u_{\hat{\delta}}\right)\right] x^{\hat{\epsilon}}x^{\hat{\delta}}x^{\hat{\gamma}}}{\hat{S}_0^{3/2}}. \tag{3.5.1}$$

We now transform to our curvilinear coordinates, $v_{\alpha} = \partial_{\alpha}x^{\hat{\mu}}v_{\hat{\mu}}$. Expanding about the position of the particle (which is the origin of both our RNC and our locally Cartesian angular coordinates), we have

$$\begin{aligned}\partial_{\alpha}x^{\hat{\mu}} &= \left(\partial_{\alpha}x^{\hat{\mu}}\right)_0 + \left(\partial_{\delta}\partial_{\alpha}x^{\hat{\mu}}\right)_0 x^{\delta} + O(x^2)\\ \partial_{\alpha}x^{\hat{\mu}} &= \left(\partial_{\alpha}x^{\hat{\mu}}\right)_0 + \left(\partial_{\epsilon}x^{\hat{\mu}}\Gamma^{\epsilon}{}_{\alpha\delta}\right)_0 x^{\delta} + O(x^2),\end{aligned} \tag{3.5.2}$$

where the subscript '0' denotes the value of a quantity at the position of the particle at time $t = 0$.

Applying this coordinate transformation, we find

$$\mathrm{A}^{S}_{\alpha} = \frac{u_{\alpha}}{\sqrt{S_0}} + \frac{\zeta_{\alpha\gamma\delta\epsilon}x^{\gamma}x^{\delta}x^{\epsilon}}{S_0^{3/2}}, \tag{3.5.3}$$

where

$$\zeta_{\alpha\gamma\delta\epsilon} := \left(2u_{\sigma}\Gamma^{\sigma}{}_{\alpha\delta} - a_{\alpha}u_{\delta}\right)q_{\gamma\epsilon} - u_{\alpha}\zeta_{\delta\gamma\epsilon}. \tag{3.5.4}$$

To calculate the regularization parameters for electromagnetism we use Eq. (2.5.10), written as

$$f^{S,\alpha}_{EM} = eu^\beta g^{\alpha\sigma}\left[\nabla_\sigma \mathbb{A}^{sing}_\beta - \nabla_\beta \mathbb{A}^{sing}_\sigma\right] = u^\beta g^{\alpha\sigma}\left[\partial_\sigma \mathbb{A}^{sing}_\beta - \partial_\beta \mathbb{A}^{sing}_\sigma\right].$$

We now calculate the value of the individual modes of $\partial \mathbb{A}^{sing}$ in the limit that the field point approaches the source (i.e. as $\epsilon \to 0$). We then write the regularization parameters for the force as a linear combination of these.

From Eq. (3.5.3), we have

$$\partial_\mu \mathbb{A}^\alpha_S = -u^\alpha \frac{\partial_\mu S_0}{S_0^3/2} + \frac{\Lambda^\alpha{}_{\mu\beta\gamma\delta\epsilon} x^\beta x^\gamma x^\delta x^\epsilon}{S_0^{5/2}}, \tag{3.5.5}$$

where

$$\Lambda^\alpha{}_{\mu\beta\gamma\delta\epsilon} = 3\zeta^\alpha{}_{(\mu\beta\gamma)} q_{\delta\epsilon} - 3\zeta^\alpha{}_{\beta\gamma\delta} q_{\mu\epsilon}. \tag{3.5.6}$$

In Eq. (3.5.5), the leading order term is simply the four-velocity multiplied by the leading order term of the scalar field. We can therefore immediately evaluate the mode decomposition of this term,

$$\begin{aligned} A^\alpha{}_\mu L &= \left[u^\alpha \frac{-\partial_\mu S_0}{S_0^3/2}\right]_\ell = u^\alpha \lim_{\delta r \to 0^\pm} \frac{L}{2\pi} \int d\cos(\theta) P_\ell(\cos(\theta)) \int d\phi \left[\frac{-\partial_\mu S_0}{S_0^3/2}\right] \\ &= u^\alpha A^{(scalar)}_\mu L = \mp \frac{L u^\alpha}{\sqrt{g_{yy}}}\left[\frac{q_{\mu r} - q_{\mu x} q_{xr}/q_{xx} - g_{\mu y} g_{yr}/g_{yy}}{\sqrt{g_{yy}\tilde{\gamma}^2 + \lambda(g_{yy} + \Gamma^2)}}\right], \end{aligned} \tag{3.5.7}$$

where we have used Eq. (3.4.42).

Now, we define

$$\Lambda^\alpha{}_{\mu XXYY} = \Lambda^\alpha{}_{\mu xxyy} + \Lambda^\alpha{}_{\mu xyxy} + \Lambda^\alpha{}_{\mu xyyx} + x \leftrightarrow y, \tag{3.5.8}$$

which we use to write (recalling $w = \beta^2(1+\beta^2)^{-1}$)

$$\begin{aligned} B^\alpha{}_\mu &= \left[\frac{\Lambda^\alpha{}_{\mu\beta\gamma\delta\epsilon} x^\beta x^\gamma x^\delta x^\epsilon}{S_0^{5/2}}\right]_\ell \\ &= \lim_{\delta r \to 0^\pm} \frac{L}{2\pi} \int d\cos(\theta) P_\ell(\cos(\theta)) \int d\phi \left[\frac{\Lambda^\alpha{}_{\mu\beta\gamma\delta\epsilon} x^\beta x^\gamma x^\delta x^\epsilon}{S_0^{5/2}}\right] \\ &= \frac{2}{3\pi(1+\beta^2)^{3/2}\beta^4 q_{yy}^{5/2}}\left(B^{(E),\alpha}_\mu \hat{E}(w) + B^{(K),\alpha}_\mu \hat{K}(w)\right), \end{aligned} \tag{3.5.9}$$

where we define

$$B^{(E),\alpha}{}_{\mu} = (1+\beta^2)(2+\beta^2)\Lambda^{\alpha}{}_{\mu XXYY} - 2\Big[(1+2\beta^2)\Lambda^{\alpha}{}_{\mu xxxx} + (1+\beta^2)^2(1-\beta^2)\Lambda^{\alpha}{}_{\mu yyyy}\Big], \tag{3.5.10}$$

and

$$B^{(K),\alpha}{}_{\mu} = (2+3\beta^2)\Lambda^{\alpha}{}_{\mu xxxx} + (1+\beta^2)\left[(2-\beta^2)\Lambda^{\alpha}{}_{\mu yyyy} - 2\Lambda^{\alpha}{}_{\mu XXYY}\right]. \tag{3.5.11}$$

We have cast Eqs. (3.5.9), (3.5.10) and (3.5.11), into forms matching those of Eqs. (3.4.25), (3.4.26a), and (3.4.26b) for the scalar case. The sole differences are the presence of the additional raised index and the additional term in the definition of $\Lambda^{\alpha}{}_{\mu\beta\gamma\delta\epsilon}$. We will see similar symmetries between the scalar field and gravity in the next section.

Now we will write down the regularization parameters in terms of $A_{\alpha\mu}$ and $B_{\alpha\mu}$.

$$f_{\alpha}^{S,EM\ell} = \left[u^{\beta}\left(\partial_{\alpha}\mathbb{A}_{\beta}^{S} - \partial_{\beta}\mathbb{A}_{\alpha}^{S}\right)\right]_{\ell}$$
$$f_{\alpha}^{S,EM\ell} = u^{\beta}\left[2A_{[\beta\alpha]}L + 2B_{[\beta\alpha]}\right]. \tag{3.5.12}$$

Restoring the factors of the charge e, we find

$$A_{\alpha}^{(EM)} = 2e^2u^{\beta}A_{[\beta\alpha]} \qquad B_{\alpha}^{(EM)} = 2e^2u^{\beta}B_{[\beta\alpha]}. \tag{3.5.13}$$

3.5.2 Gravitational Regularization Parameters

From Eq. (2.5.28), we can write the singular part of the trace-reversed metric perturbation as

$$\gamma_{\hat{\alpha}\hat{\beta}}^{S} = \frac{4u_{\hat{\alpha}}u_{\hat{\beta}}}{\sqrt{\hat{S}_0}} - 4\frac{\left[2u_{(\hat{\alpha}}a_{\hat{\beta})}u_{\hat{\epsilon}}q_{\hat{\delta}\hat{\gamma}} + u_{\hat{\alpha}}u_{\hat{\beta}}\zeta_{\hat{\epsilon}\hat{\delta}\hat{\gamma}}\right]x^{\hat{\epsilon}}x^{\hat{\delta}}x^{\hat{\gamma}}}{\hat{S}_0^{3/2}}. \tag{3.5.14}$$

We write this in terms of the actual metric perturbation, $h_{\mu\nu} = \gamma_{\mu\nu} - 1/2g_{\mu\nu}\gamma^{\mu}_{\mu}$, and then apply the coordinate transformation to take us from RNCs to our curvilinear coordinates. Upon doing this, we find,

$$h_S^{\alpha\beta} = 2\frac{g^{\alpha\beta} + 2u^{\alpha}u^{\beta}}{\sqrt{S_0}} + \frac{\zeta^{\alpha\beta}{}_{\gamma\delta\epsilon}x^{\gamma}x^{\delta}x^{\epsilon}}{S_0^{3/2}}, \tag{3.5.15}$$

where

$$\zeta^{\alpha\beta}{}_{\gamma\delta\epsilon} := \left(8u^{(\alpha}a^{\beta)}u_\gamma - \partial_\gamma g^{\alpha\beta} + 4u^\sigma u^{(\alpha}\Gamma_\sigma{}^{\beta)}{}_\gamma\right) q_{\delta\epsilon} + (g^{\alpha\beta} + 2u^\alpha u^\beta)\zeta_{\gamma\delta\epsilon}. \tag{3.5.16}$$

We now compute $f^{\alpha,S}_{GR}$ from Eq. (2.5.26),

$$\begin{aligned} f^{\alpha,S}_{GR} &= -m\, q^{\alpha\delta}\left(\nabla_\beta h^{(s)}_{\gamma\delta} - \frac{1}{2}\nabla_\delta h^{(s)}_{\beta\gamma}\right) u^\beta u^\gamma \\ &= -m\left(g^{\alpha\delta} + u^\alpha u^\delta\right) u^\beta u^\gamma \left(\partial_\beta h^S_{\gamma\delta} - \frac{1}{2}\partial_\delta h^S_{\beta\gamma} - \Gamma^\mu{}_{\beta\gamma} h^S_{\mu\delta} + \Gamma^\mu{}_{\delta[\gamma} h^S_{\beta]\mu}\right). \end{aligned} \tag{3.5.17}$$

Therefore, we need to find the leading terms in the mode-sum decomposition of the metric perturbation and its derivative.

We first discuss the mode sum decomposition of the metric perturbation itself. Because the sub-leading term, is cubic in the coordinates $x^m u$ and is $O(\epsilon^0)$, its contribution will vanish. This means that the mode-sum decomposition of the metric perturbation evaluated at the position of the mass at time $t = 0$, is given by

$$\begin{aligned} h^{\alpha\beta}_{S,\ell} &= 2 \lim_{\delta r \to 0^\pm} \frac{L}{2\pi} \int d\cos(\theta) P_\ell(\cos(\theta)) \int d\phi \left[\frac{g^{\alpha\beta} + 2u^\alpha u^\beta}{\sqrt{S_0}}\right] \\ h^{\alpha\beta}_{S,\ell} &= B^{\alpha\beta}_{(h)} = 2\left(g^{\alpha\beta} + 2u^\alpha u^\beta\right)\left[\frac{2}{\pi(1+\beta^2)^{1/2}}\hat{K}(w)\right]. \end{aligned} \tag{3.5.18}$$

We use the subscript, (h) to distinguish $B^{\alpha\beta}_{(h)}$ from the quantity $B^{\alpha\beta}$ of the electromagnetism section above.

From Eq. (3.5.15), we have

$$\partial_\mu h^{\alpha\beta}_S = -(g^{\alpha\beta} + 2u^\alpha u^\beta)\frac{\partial_\mu S_0}{2S_0^{3/2}} + \frac{\Lambda^{\alpha\beta}{}_{\mu\gamma\delta\epsilon\sigma} x^\gamma x^\delta x^\epsilon x^\sigma}{S_0^{5/2}}, \tag{3.5.19}$$

where

$$\Lambda^{\alpha\beta}{}_{\mu\gamma\delta\epsilon\sigma} := \left[3\zeta^{\alpha\beta}{}_{(\mu\gamma\delta)} q_{\epsilon\sigma} - 3\zeta^{\alpha\beta}{}_{\gamma\delta\epsilon} q_{\mu\sigma}\right]. \tag{3.5.20}$$

The leading order term has the form

$$\begin{aligned} A^{\alpha\beta}{}_\mu L &= \left[-(g^{\alpha\beta}+2u^\alpha u^\beta)\frac{\partial_\mu S_0}{2S_0^{3/2}}\right]_\ell \\ &= -(g^{\alpha\beta}+2u^\alpha u^\beta)\lim_{\delta r\to 0^\pm}\frac{L}{2\pi}\int d\cos(\theta)P_\ell(\cos(\theta))\int d\phi\left[\frac{\partial_\mu S_0}{2S_0^{3/2}}\right] \\ &= (g^{\alpha\beta}+2u^\alpha u^\beta)A_\mu^{(scalar)}L = A^{\alpha\beta}{}_\mu L \\ &= \mp\frac{L(g^{\alpha\beta}+2u^\alpha u^\beta)}{\sqrt{g_{yy}}}\left[\frac{g_{\mu r}+u_\mu u_r-\frac{(g_{\mu x}+u_\mu u_x)(g_{xr}+u_x u_r)}{g_{xx}+U_x^2}-\frac{g_{\mu y}g_{yr}}{g_{yy}}}{\sqrt{g_{yy}\tilde{\gamma}^2+\lambda(g_{yy}+\Gamma^2)}}\right]. \end{aligned} \tag{3.5.21}$$

Now, we define

$$\Lambda^{\alpha\beta}{}_{\mu XXYY} = \Lambda^{\alpha\beta}{}_{\mu xxyy} + \Lambda^{\alpha}{}_{\mu xyxy} + \Lambda^{\alpha}{}_{\mu xyyx} + x \leftrightarrow y, \tag{3.5.22}$$

which allows us to write, (recalling $w = \beta^2(1+\beta^2)^{-1}$)

$$\begin{aligned} B^{\alpha\beta}{}_\mu &= \left[\frac{\Lambda^{\alpha\beta}{}_{\mu\sigma\gamma\delta\epsilon}x^\sigma x^\gamma x^\delta x^\epsilon}{S_0^{5/2}}\right]_\ell \\ &= \lim_{\delta r\to 0^\pm}\frac{L}{2\pi}\int d\cos(\theta)P_\ell(\cos(\theta))\int d\phi\left[\frac{\Lambda^{\alpha\beta}{}_{\mu\sigma\gamma\delta\epsilon}x^\sigma x^\gamma x^\delta x^\epsilon}{S_0^{5/2}}\right] \\ &= \frac{2}{3\pi(1+\beta^2)^{3/2}\beta^4 q_{yy}^{5/2}}\left(B^{(E),\alpha\beta}{}_\mu\hat{E}(w)+B^{(K),\alpha\beta}{}_\mu\hat{K}(w)\right), \end{aligned} \tag{3.5.23}$$

where we define

$$\begin{aligned} B^{(E),\alpha\beta}{}_\mu = -2\left[(1+2\beta^2)\Lambda^{\alpha\beta}{}_{\mu xxxx}+(1+\beta^2)^2(1-\beta^2)\Lambda^{\alpha\beta}{}_{\mu yyyy}\right] \\ +(1+\beta^2)(2+\beta^2)\Lambda^{\alpha\beta}{}_{\mu XXYY}, \end{aligned} \tag{3.5.24}$$

and

$$B^{(K),\alpha\beta}{}_\mu = (1+\beta^2)\left[(2-\beta^2)\Lambda^{\alpha\beta}{}_{\mu yyyy}-2\Lambda^{\alpha\beta}{}_{\mu XXYY}\right]+(2+3\beta^2)\Lambda^{\alpha\beta}{}_{\mu xxxx}. \tag{3.5.25}$$

We can now write the regularization parameters for gravity. From Eqs. (3.5.17), (3.5.18), (3.5.21), and (3.5.23), we see that only the partial derivatives of the metric perturbation contribute to $A^\alpha_{(GR)}$, allowing us to write,

$$A^\alpha_{(GR)} = -m\left(g^{\alpha\delta}+u^\alpha u^\delta\right)u^\beta u^\gamma\left(A_{\gamma\delta\beta}-\frac{1}{2}A_{\beta\gamma\delta}\right). \tag{3.5.26}$$

The components $B^{\alpha}_{(GR)}$ are given by

$$B^{\alpha}_{(GR)} = -m\left(g^{\alpha\delta} + u^{\alpha}u^{\delta}\right)u^{\beta}u^{\gamma}\left(B_{\gamma\delta\beta} - \frac{1}{2}B_{\beta\gamma\delta} + \Gamma^{\mu}{}_{\delta[\gamma}B^{(h)}_{\beta]\mu} - \Gamma^{\mu}{}_{\beta\gamma}B^{(h)}_{\mu\delta}\right). \qquad (3.5.27)$$

We have obtained the explicit forms of the regularization parameters for all three spins in Eqs. (3.4.42) and (3.4.25) (scalar); (3.5.13) (electromagnetism); and (3.5.26) and (3.5.27) (gravity). For all three spins, we have given the values in terms of ζ coefficients, which represent the numerator of the sub-leading terms of the potential (or perturbing metric), and Λ coefficients, which represent the numerator of the sub leading terms of the derivative of the potential (or perturbing metric).

3.6 Regularization Parameters in the Original Background Coordinates

In Sects. 3.4 and 3.5, the components of the regularization parameters are obtained along a basis associated with locally Cartesian angular coordinates (LCAC); and the value we obtain for the vector B_{α} relies on extending the components of $q_{\alpha\beta}$ and u^{α} away from the particle by requiring that their components in the LCAC basis assume the values they take at the particle. For many applications, it is more useful to evaluate the components of A_{α} and B_{α} in the original coordinate system, as first done by Barack and Ori [44] and then later explained more completely in an appendix by Barack [42]. In this section, we follow the latter treatment and freeze the components of u^{α} and $q_{\alpha\beta}$ in the original t, r, θ, ϕ coordinates.

We define $(\tilde{x}^{\alpha}) = (\delta t = t, \delta r = r - r_0, \delta\theta = \theta - \theta_0, \delta\phi = \phi - \phi_0)$, so that $\tilde{x}^{\mu}$ agrees up to a constant with the original $t, r\theta, \phi$ coordinates; we continue to denote the locally Cartesian coordinates by $x^{\alpha} = (\delta t, \delta r, x, y)$. We denote by $\widetilde{W}^{\mu\ldots\nu}_{\sigma\ldots\tau}$ the components of a quantity $W^{\cdots}_{\cdots}$, evaluated using the coordinate system x^{μ}. Note that the quantities $\zeta_{\mu\nu\lambda}$ and $\Lambda_{\mu\ldots\nu}$ involve partial derivatives of metric components and do not transform as tensors.

From the definitions of S_0, S_1, and the derivative of our singular field, (Eqs. (3.4.10), (3.4.11), and (2.2.34) respectively), we can write the components of the singular force in the original coordinates as

$$q^{-2}\tilde{f}^S_\mu = -\frac{\tilde{q}_{\mu\nu}\tilde{x}^\nu}{\widetilde{S}_0^{3/2}} + \frac{3\tilde{\zeta}_{\gamma\delta\epsilon}q_{\mu\nu} - (2\tilde{\zeta}_{\mu\gamma\delta} + \tilde{\zeta}_{\gamma\delta\mu})\tilde{q}_{\nu\epsilon}}{\widetilde{S}_0^{5/2}}\tilde{x}^\nu\tilde{x}^\epsilon\tilde{x}^\gamma\tilde{x}^\delta + O(\epsilon^0). \tag{3.6.1}$$

We still want to use the LCAC to simplify our integrations, retaining the $\tilde{x}^\mu$ components $\widetilde{W}^{\mu\ldots\nu}_{\sigma\ldots\tau}$ of each quantity, but expressing them in terms of the LCAC. To do so, we write

$$\begin{aligned}\tilde{x}^3 &= \delta\theta = x^3 + \frac{1}{2}\cot(\theta_0)(x^4)^2 + O(\epsilon^3)\\ \tilde{x}^4 &= \delta\phi = \sin(\theta_0)^{-1}\left(x^4 - \cot(\theta_0)x^3x^4\right) + O(\epsilon^3)\end{aligned} \tag{3.6.2}$$

(equivalent to Eq. (A.17) of [42]). Then

$$\tilde{x}^\alpha = a^\alpha_\beta x^\beta + c^\alpha_{\beta\gamma}x^\beta x^\gamma + O(\epsilon^3), \tag{3.6.3}$$

where $a^\alpha_\beta = \partial_\beta\tilde{x}^\alpha|_0$, and $c^\alpha_{\beta\gamma} = \partial_\beta\partial_\gamma\tilde{x}^\alpha|_0$. By the arguments laid down before, it is clear that the higher order terms will give contributions to the self-force that either vanish at the particle or contribute to an order-unity term that vanishes upon integration over ϕ. Note that, at linear order, the transformation (3.6.3) just replaces each occurrence of $\tilde{x}^4$ by $x^4/\sin\theta_0$.

The leading term acquires a first order correction:

$$\tilde{f}^{S,L}_\mu = -\frac{\tilde{q}_{\mu\nu}a^\nu_\lambda x^\lambda}{(\tilde{q}_{\alpha\beta}a^\alpha_\sigma a^\beta_\tau x^\sigma x^\tau)^{3/2}} + \frac{(3\tilde{q}_{\mu\nu}\tilde{q}_{\iota\kappa} - \tilde{q}_{\mu\iota}\tilde{q}_{\nu\kappa})\,c^\iota_{\sigma\tau}x^\nu x^\kappa x^\sigma x^\tau}{(\tilde{q}_{\alpha\beta}a^\alpha_\sigma a^\beta_\tau x^\sigma x^\tau)^{5/2}} \tag{3.6.4}$$

We take the mode-sum expansion of the force and evaluate these individual modes in the limit that $\epsilon \to 0$. The leading term will now give us the A_α term as before, and in the original coordinates we merely pick up an additional factor of $\sin\theta_0$;

$$\widetilde{A}_{\alpha\pm} = \mp\sin\theta_0\; q^2\,\frac{\tilde{q}_{\alpha r} - \tilde{q}_{\alpha\theta}\tilde{q}_{\theta r}/\tilde{q}_{\theta\theta} - \tilde{q}_{\alpha\phi}\tilde{q}_{\phi r}/\tilde{q}_{\phi\phi}}{(\tilde{q}_{\theta\theta}\tilde{q}_{\phi\phi}\tilde{q}_{rr} - \tilde{q}_{\phi\phi}\tilde{q}^2_{\theta r} - \tilde{q}_{\theta\theta}\tilde{q}^2_{\phi r})^{1/2}}. \tag{3.6.5}$$

For $\widetilde{B}_\alpha$, we evaluate the integral

$$\widetilde{B}_\alpha = \frac{q^2}{2\pi}\widetilde{P}_{\alpha\mu\nu\sigma\tau}\widetilde{I}^{\mu\nu\sigma\tau}, \tag{3.6.6}$$

where

$$\widetilde{I}^{\mu\nu\sigma\tau} = \lim_{\delta r\to 0}\int_0^{2\pi} d\phi\left[\frac{a^\mu_\alpha a^\nu_\beta a^\sigma_\gamma a^\tau_\delta x^\alpha x^\beta x^\gamma x^\delta}{(\tilde{q}_{\kappa\lambda}a^\kappa_\epsilon a^\lambda_\iota x^\epsilon x^\iota)^{5/2}}\right], \tag{3.6.7}$$

and

$$\widetilde{P}_{\alpha\mu\nu\gamma\delta} = 3\tilde{q}_{\alpha\delta}\tilde{\zeta}_{\mu\nu\gamma} - \tilde{q}_{\gamma\delta}\left(2\tilde{\zeta}_{\alpha\mu\nu} + \tilde{\zeta}_{\mu\nu\alpha}\right) + \left(3\tilde{q}_{\alpha\mu}\tilde{q}_{\epsilon\nu} - \tilde{q}_{\alpha\epsilon}\tilde{q}_{\mu\nu}\right) c^{\epsilon}{}_{\gamma\delta}, \quad (3.6.8)$$

where $c^{\epsilon}{}_{\gamma\delta}$ is defined in Eq. (3.6.3), whose only non-vanishing components are $c^{\theta}{}_{\phi\phi} = 4^{-1}\sin(2\theta_0)$ and $c^{\phi}{}_{\theta\phi} = c^{\phi}{}_{\phi\theta} = -2^{-1}\cot(\theta_0)$.

Notice that this equation is identical to Eq. (58) from [42], with the sole exception that we have included the acceleration in our $\tilde{\zeta}_{\alpha\beta\gamma}$. The limit in Eq. (3.6.7) means that the integral $I^{\mu\nu\gamma\delta}$ vanishes except when the indices only run over the (θ, ϕ) coordinates. Adopting the notation from [42], we let lowercase roman indices run over only θ and ϕ. Barack writes down the solutions to these integrals in Eqs. (48-57) [42], which we reproduce below. First, we define

$$\alpha = \sin^2(\theta_0)\tilde{q}_{\theta\theta}/\tilde{q}_{\phi\phi} - 1, \qquad \tilde{\beta} = 2\sin(\theta_0)\tilde{q}_{\theta\phi}/\tilde{q}_{\phi\phi}. \quad (3.6.9)$$

Then, I^{abcd} is given by

$$I^{abcd} = \frac{\sin(\theta_0)^{5-N}}{(\alpha^2+\tilde{\beta}^2)^2(4\alpha+4-\tilde{\beta}^2)^{3/2}(Q/2)^{1/2}}\left[Q I_K^{(N)}\hat{K}(\omega) + I_E^{(N)}\hat{E}(\omega)\right], \quad (3.6.10)$$

where

$$Q = \alpha + 2 - (\alpha^2+\tilde{\beta}^2)^{1/2}, \qquad \omega = \frac{2(\alpha^2+\tilde{\beta}^2)^{1/2}}{\alpha+2+(\alpha^2+\tilde{\beta}^2)^{1/2}}, \quad (3.6.11)$$

and $N = \delta^a_\phi + \delta^b_\phi + \delta^c_\phi + \delta^d_\phi$.

The ten quantities $I_K^{(N)}$ and $I_E^{(N)}$ are given by

$$\begin{aligned} I_K^{(0)} &= 4\left[12\alpha^3 + \alpha^2(8-3\tilde{\beta}^2) - 4\alpha\tilde{\beta}^2 + \tilde{\beta}^2(\tilde{\beta}^2-8)\right], \\ I_E^{(0)} &= -16\left[8\alpha^3 + \alpha^2(4-7\tilde{\beta}^2) + \alpha\tilde{\beta}^2(\tilde{\beta}^2-4) - \tilde{\beta}^2(\tilde{\beta}^2+4)\right], \end{aligned} \quad (3.6.12)$$

$$\begin{aligned} I_K^{(1)} &= 8\tilde{\beta}\left[9\alpha^2 - 2\alpha(\tilde{\beta}^2-4) + \tilde{\beta}^2\right], \\ I_E^{(1)} &= -4\tilde{\beta}\left[12\alpha^3 - \alpha^2(\tilde{\beta}^2-52) + \alpha(32-12\tilde{\beta}^2) + \tilde{\beta}^2(3\tilde{\beta}^2+4)\right], \end{aligned} \quad (3.6.13)$$

$$\begin{aligned} I_K^{(2)} &= -4\left[8\alpha^3 - \alpha^2(\tilde{\beta}^2-8) - 8\alpha\tilde{\beta}^2 + \tilde{\beta}^2(3\tilde{\beta}^2-8)\right], \\ I_E^{(2)} &= 8\left[4\alpha^4 + \alpha^3(\tilde{\beta}^2+12) + \alpha(\tilde{\beta}^2-4)(3\tilde{\beta}^2-2\alpha) + 2\tilde{\beta}^2(3\tilde{\beta}^2-4)\right], \end{aligned} \quad (3.6.14)$$

$$
\begin{aligned}
I_K^{(3)} &= 8\tilde{\beta}\left[\alpha^3 - 7\alpha^2 + \alpha(3\tilde{\beta}^2 - 8) + \tilde{\beta}^2\right], \\
I_E^{(3)} &= -4\tilde{\beta}\left[8\alpha^4 - 4\alpha^3 + \alpha^2(15\tilde{\beta}^2 - 44) + 4\alpha(5\tilde{\beta}^2 - 8) + \tilde{\beta}^2(3\tilde{\beta}^2 + 4)\right],
\end{aligned}
\tag{3.6.15}
$$

$$
\begin{aligned}
I_K^{(4)} &= -4\left[4\alpha^4 - 4\alpha^3 + \alpha^2(7\tilde{\beta}^2 - 8) + 12\alpha\tilde{\beta}^2 - \tilde{\beta}^2(\tilde{\beta}^2 - 8)\right], \\
I_E^{(4)} &= 16\left[4\alpha^5 + 4\alpha^4 + \alpha^3(7\tilde{\beta}^2 - 4) + \alpha^2(11\tilde{\beta}^2 - 4) + (2\alpha + 1)\tilde{\beta}^2(\tilde{\beta}^2 + 4)\right].
\end{aligned}
\tag{3.6.16}
$$

3.6.1 The Regularization Parameters for Electromagnetism and Gravity

First, recall Eq. (3.6.22), reproduced below:

$$
\begin{aligned}
f_\mu^{s=1,S} &= \left(\delta_\mu^\beta u^\alpha - \delta_\mu^\alpha u^\beta\right)\nabla_\beta A_\alpha^S \\
f_\mu^{s=2,S} &= \left(q_\mu^\beta\left(q^{\gamma\delta} + u^\gamma u^\delta\right) - 4q_\mu^\delta u^\beta u^\gamma\right)\nabla_\beta \frac{\gamma_{\gamma\delta}^S}{4}.
\end{aligned}
$$

Since we have shown that only the leading and subleading terms in the singular vector potential and metric perturbation will give a non-vanishing contribution to the mode-sum when evaluated at the particle, this allows us to write the expressions for the singular vector potential and metric perturbation in a very convenient form, (taking the charge and mass to be unity)

$$
\begin{aligned}
\mathrm{A}_{\hat{\alpha}}^S &= \left[u_{\hat{\alpha}} - a_{\hat{\alpha}} u_\nu x^\nu + O(\epsilon^1)\right]\Phi^S \\
\frac{1}{4}\gamma_{\hat{\alpha}\hat{\beta}}^S &= \left[u_{\hat{\alpha}} u_{\hat{\beta}} - 2a_{(\hat{\alpha}} u_{\hat{\beta})} u_\nu x^\nu + O(\epsilon^1)\right]\Phi^S.
\end{aligned}
\tag{3.6.17}
$$

We transform from the RNC basis to the coordinate basis using Eq. (3.4.9), and plug in our expression for $\Phi^S = S_0^{-1/2} - S_1(2S_0^{3/2})^{-1} + O(\epsilon^1)$, we find that the singular force for spin $s = 0, 1, 2$ can be written as

$$
\tilde{f}_\alpha^{s,S} = (-1)^s (q_s)^2 \left[-\frac{\tilde{q}_{\alpha\nu}\tilde{x}^\nu}{\tilde{S}_0^{3/2}} + \frac{\tilde{P}^s_{\alpha\mu\nu\gamma\delta}\tilde{x}^\mu\tilde{x}^\nu\tilde{x}^\gamma\tilde{x}^\delta}{\tilde{S}_0^{5/2}} + O(\epsilon^0)\right],
\tag{3.6.18}
$$

where q_s is q, e, m for $s = 0, 1, 2$ respectively, and $P^s_{\alpha\mu\nu\gamma\delta}$ is given by

$$
\tilde{P}^s_{\alpha\mu\nu\gamma\delta} = \left(\delta_{s,0}\delta_\alpha^\beta + \tilde{q}_\alpha^\beta(1 - \delta_{s,0})\right)\left(\tilde{P}_{\beta\mu\nu\gamma\delta} + s^2\tilde{a}_\beta\tilde{q}_{\mu\nu}\tilde{q}_{\gamma\delta} + s\tilde{q}_{\beta\gamma}\tilde{u}^\lambda\tilde{u}^\rho\partial_\delta\tilde{g}_{\lambda\rho}\tilde{q}_{\mu\nu}\right),
\tag{3.6.19}
$$

where $P_{\beta\mu\nu\gamma\delta}$ is defined in Eq. (3.6.8). Thus, we can write the regularization parameters for spins 0,1, and 2:

$$\tilde{A}^s_{\alpha\pm} = \mp \sin(\theta_0) q_s^2 (-1)^s \frac{\tilde{q}_{\alpha r} - \tilde{q}_{\alpha\theta}\tilde{q}_{\theta r}/\tilde{q}_{\theta\theta} - \tilde{q}_{\alpha\phi}\tilde{q}_{\phi r}/\tilde{q}_{\phi\phi}}{(\tilde{q}_{\theta\theta}\tilde{q}_{\phi\phi}\tilde{q}_{rr} - \tilde{q}_{\phi\phi}\tilde{q}^2_{\theta r} - \tilde{q}_{\theta\theta}\tilde{q}^2_{\phi r})^{1/2}}, \tag{3.6.20}$$

and

$$\tilde{B}^s_\alpha = (-1)^s \frac{q_s^2}{2\pi} \tilde{P}^s_{\alpha\mu\nu\gamma\delta} \tilde{I}^{\mu\nu\gamma\delta}, \tag{3.6.21}$$

where $I^{\mu\nu\gamma\delta}$ is given in Eq. (3.6.7).

Eqs. (3.6.20) and (3.6.21) simplify exactly to Eqs. (39-44) given in [42], when we take the geodesic limit, and specialize to a Kerr geometry.

3.6.2 Extending quantities away from the world line

The expressions for the self-force in an electromagnetic or gravitational context depend on how one extends $g^{\alpha\beta}[z(0)]$ and $u^\alpha[z(0)]$ to a neighborhood of the particle (and there is even this ambiguity in how one defines the scalar self-force with raised indices). If we return to the definition of the scalar, electromagnetic, or gravitational self-force, (Eqs. (3.2.2), (2.5.10) or (2.5.26), then we can rewrite them as

$$\begin{aligned} f^{s=0,S,\mu} &= k^{\mu\nu}\nabla_\nu \Phi^{sing} = g^{\mu\nu}\nabla_\nu \Phi^{sing} \\ f^{s=1,S,\mu} &= k^{\mu\alpha\beta}\nabla_\beta \mathrm{A}^S_\alpha = \left(\delta^{\mu\beta}u^\alpha - \delta^{\mu\alpha}u^\beta\right)\nabla_\beta \mathrm{A}^S_\alpha \\ f^{s=2,S,\mu} &= k^{\mu\beta\gamma\delta}\nabla_\beta \gamma^S_{\gamma\delta} = \frac{\left(q^{\beta\mu}\left(q^{\gamma\delta} + u^\gamma u^\delta\right) - 4q^{\delta\mu}u^\beta u^\gamma\right)}{4}\nabla_\beta \gamma^S_{\gamma\delta}. \end{aligned} \tag{3.6.22}$$

In particular, the quantities $k^{\mu\cdots}$ are only properly defined on the trajectory of the particle for $s = 1, 2$, and we are allowed a choice in how we extend $k^{\mu\cdots}$ away from the world line. One popular way is to use the 'fixed extension' [42], in which one defines $k^{\mu\cdots}(x \neq z(0)) = k^{\mu\cdots}(x = z(0))$, and is the one we use in this paper, but other choices are available [22]. We now show that as long as $k^{\mu\cdots}$ is a smooth function in x then the regularization parameters retain the form $A_\alpha L + B_\alpha$.

Since each component of A^S_α and $\gamma^S_{\alpha\beta}$ has the same algebraic form as Φ^S, we will consider finding the regularization parameters for $f^{s=0,S,\mu}$. Denote by $k_0^{\mu\nu}$, $\partial_\gamma k_0^{\mu\nu}$, and

$\partial_\delta\partial_\gamma k_0^{\mu\nu}$ the values of $k^{\mu\nu}$ and its derivatives at $z(0)$. For an extension $k^{\mu\nu}[x]$ of $k^{\mu\nu}[z(0)]$ the departure of $k^{\mu\nu}\nabla_\nu\Phi^S$ from $k_0^{\mu\nu}\nabla_\nu\Phi^S$ is given by

$$(k^{\mu\nu} - k_0^{\mu\nu})\nabla_\nu\Phi^S = x^\gamma\partial_k^{\mu\nu} g_0 \nabla_\nu\Phi^{S,L} + \left(x^\gamma\partial_\gamma k_0^{\mu\nu}\nabla_\nu\Phi^{S,SL} + \frac{1}{2}x^\gamma x^\delta\partial_\gamma\partial_\delta k^{\mu\nu}\nabla_\nu\Phi^{S,L}\right) + O(\epsilon). \tag{3.6.23}$$

The first term on the right has the form $P^{(4)}(x^\mu)S_0^{-5/2}$, and it thus gives a correction to the B term. The term in parentheses on the right is order unity and has the form $P^{(7)}(x^\mu)S_0^{-7/2}$; its contribution to the $f^{SSL,\ell}$, given by its contribution to the integral on the right side of Eq. (3.4.13) therefore vanishes. Because the remaining part of the right side of (3.6.23) is $O(\epsilon)$, its contribution to the f_α^S also vanishes.

Therefore, we have demonstrated our claim in Eq. (3.4.3). In doing so, we have shown that to regularize the fields themselves, one needs only subtract of a 'B' term from the mode-sum of the retarded field, which is to say, for a field ψ, $\psi_{\ldots}^{S,\ell} = B_{\ldots}$.

3.7 Discussion

By moving into the basis of spherical harmonics and analyzing quantities mode by mode, it can be difficult to connect the results we find to the physics we are trying to model, so it is useful to gain an appreciation for the similarities in the mode-sum prescription and the *MiSaTaQuWa* and Detweiler and Whiting prescriptions.

Our key tool for making these comparisons is the insight we already mentioned, namely that the singular behavior of the fields is uniquely determined by the high ℓ behavior of the harmonic modes. In Chapter 2, we used the idea that the singular nature of the fields is uniquely determined by the small ϵ behavior of the fields.

When we regularized in Chapter 2, we evaluated the fields at a small, but finite distance ϵ away from $z(0)$. This, then, is identical to truncating our expression in the harmonics by evaluating only to a maximum ℓ mode, $\ell = \ell_{max}$. Then to renormalize, we take the limit as $\epsilon \to 0$ or as $\ell_{max} \to \infty$.

Surprisingly, we can push these analogies even further, and by doing so we can gain an appreciation for the *practical* difficulties that still remain after performing the mode-sum renormalization. Recall in Chapter 2, where we discussed the difficulties arising from the angle-average demanded by the *MiSaTaQuWa* prescription. In this prescription, we subtracted away the flat spacetime field and performed the angle-average to get rid of the finite but direction-dependent terms from the sub-subleading terms. This is an elegant way of presenting the procedure, although the angle-average itself is frequently impractical (perhaps prohibitively so) to implement.

By pushing this analogy further, we can understand the origin of the remaining *pracitical* difficulties in mode-sum renormalization. In the mode-sum scheme, we have just demonstrated that the renormalization can be performed by merely subtracting the A_α and B_α terms. Therefore, it would be tempting to say that

$$f_\alpha^{R,\ell} = f_\alpha^{ret,\ell} - (\pm A_\alpha L + B_\alpha)\,, \tag{3.7.1}$$

and, indeed, if we make this definition and then perform the sum over *all* modes $f_\alpha^R = \sum_{\ell=0}^{\infty} f_\alpha^{R,\ell}$, we would indeed find the renormalized force.

On the other hand, earlier we stated that we would treat the field Φ^R as a C^∞ function, which means that its ℓ modes should fall off faster than any power of ℓ, so we should be able to get a very accurate expression by simply keeping the first handful of modes. But when we consider results from numerical and analytic work, it is clear that the modes defined in Eq. (3.7.1) fall off only as ℓ^{-2}.

This would seem to be a contradiction. After all, we computed the regularization parameters by taking the mode-sum decomposition of the full Detweiler-Whiting singular field, Φ^S, and when we computed these modes we found that the only terms that arise include one term linear in ℓ and one term independent of ℓ (the A_α and B_α terms respectively).

To resolve this apparent contradiction, let us approach this from the other direction. By defining the modes of the singular force to be $\pm A_\alpha L + B_\alpha$, we are looking at only the leading and subleading contributions in ℓ. In fact, we demonstrated that these terms arise by computing only the contributions from the leading and sub-leading terms in the small ϵ expansion, which come from the flat spacetime singular field only.

Therefore, when we subtract the A_α and B_α terms, we are in fact following the original *MiSaTaQuWa* prescription, and not making use of the insights from Detweiler and

Whiting. The practical problem, as we have stated on multiple occasions, with the *MiSaTaQuWa* formulation is that we have to perform the angle-average. The difficulties associated with this angle-average translate over to the mode-sum calculations in the form of the slow convergence of the sum over ℓ.

When we take the mode-sum decomposition of $\nabla_\alpha \Phi^S$, and write

$$f_\alpha^{S,\ell} = \frac{L}{2\pi} \int d\Omega P_\ell(\cos(\theta)) \nabla_\alpha \left(\Phi^{\rm L} + \Phi^{\rm SL} + \Phi^{\rm SSL} + ...\right), \tag{3.7.2}$$

and we truncate the expression after the sub-subleading order, we include only those terms that do not vanish when $\epsilon \to 0$. But, if we only keep the terms that do not vanish as $\epsilon \to 0$, this implies we are keeping the terms that do not vanish only as we let $\ell_{max} \to \infty$. The work we have done up to this point ensures that the sum over ℓ, from $\ell = 0$ to ∞ of any of the higher order terms vanishes, but it does not give us any information on the behavior of the individual modes of these terms.

However, from our general arguments, one would expect that $\nabla_\alpha \Phi^{\rm SSSL}$ falls off as L^{-2}, and that each successive mode will fall off as successively higher powers in L^{-1}. This has been confirmed by Heffernan et al. [31], who has computed the next handful of regularization parameters in Schwarzschild and Kerr. We will not review her results in any detail here, but only remark that by following the same arguments we used to demonstrate that the D_α term vanishes, one can similarly show that the $L^{-(2n+1)}$ terms will vanish, which is what allows us to write Eq. (3.1.3), which says that

$$D_\alpha = \sum_{\ell=0}^{\infty} \sum_{n=1}^{\infty} \frac{D_\alpha^{(2n)}}{L^{2n}}.$$

Because we know that these terms must actually vanish when summed from $\ell = 0$ to $\ell = \infty$, we can actually rewrite this definition in a more appropriate form. For example, if we were to subtract a handful of these $D^{(2n)}$ terms, and consider the sum over ℓ, we would find that the sum converges much more quickly, *but it would converge to the wrong answer*. To understand this, consider the fact that

$$\sum_{\ell=0}^{\infty} \frac{1}{L^{2n}} = (2^{2n} - 1)\zeta(2n) \neq 0, \tag{3.7.3}$$

where $\zeta(x)$ is the Riemann Zeta function. Therefore subtracting just the higher order regularization parameters would introduce a systematic error in the calculation of the self-force. In practice, this can be easily remedied by simply computing the value of the sum of the higher order terms from $\ell = 0$ to $\ell = \ell_{max}$ and subtracting away this value so as to get the correct self-force.

We will slightly alter the definition of the D_α term, by changing it so that each individual expression actually will vanish when summed over all ℓ. We will use

$$\begin{aligned} D_\alpha &= \sum_{\ell=0}^{\infty}\sum_{n=1}^{\infty} \frac{4^n D_\alpha^{(2n)}}{\prod_{k=1}^{n}[(2\ell+1+2k)(2\ell+1-2k)]} \\ D_\alpha &= \sum_{\ell=0}^{\infty}\sum_{n=1}^{\infty} \frac{4^n D_\alpha^{(2n)}}{\prod_{k=1}^{n}\{k\}}, \end{aligned} \tag{3.7.4}$$

where we define $\{k\} := [(2\ell+1+2k)(2\ell+1-2k)]$.[7] This will not change the value of the D_α term–it still vanishes, but by writing it in this form the sum over each of the $D^{(2n)}$ term manifestly vanishes (see proof below). Furthermore, we have introduce the 4^n term so that the values of these modified higher order parameters will have the same values as those found by examining the large ℓ behavior.

In some respects, this altered definition is completely cosmetic. It will not change anything in the way we define the higher order parameters, and for any work where $\ell_{max} < \infty$, it will still be necessary to sum each term from $\ell = 0$ to $\ell = \ell_{max}$, so that it is possible to account for any finite contribution we would be adding by introducing the higher order regularization parameters.

On the other hand, writing the D_α term in this form will be useful when we can look at an analytic expression for the general-ℓ expression for the retarded field in Chapter 6. There, we will be able to renormalize the fields "by eye," by writing terms that fall off as ℓ to a finite power in terms of this finite sum in addition to a term that falls off faster than any power of ℓ, allowing us to pick out pieces of the singular and renormalized field on sight.

So, let us return to the original contradiction, where it seemed that the function ϕ^R that we were treating as a C^∞ function fell off only as ℓ^{-2}. By including the additional $D_\alpha^{(2n)}$ terms we increase the rate of convergence by a factor of ℓ^{-2} for each term we include. In principle we could continue to compute the $D_\alpha^{(2n)}$ terms to arbitrarily high order [8] so

[7] The from in the first line of Eq. (3.7.4) was first used by Detweiler et al. [48]

[8] although the practical difficulties become prohibitive after the first several terms

by adding these terms we can make the field fall off as quickly as we like.

Similarly, including successively more terms in the mode-sum is akin to including successively more terms in the local expansion of the singular field. Therefore, as we can see, the ambiguity in the definition of the singular field discussed in section 2.4 works to our advantage as it allows us to make Φ^R as smooth as we like at the particle.

Now, we will give the proof that these sums actually do vanish.

3.7.1 Vanishing Sums

We show the relation [9]

$$\sum_{\ell=0}^{\infty}\prod_{j=0}^{N}\frac{1}{(2\ell+1-2m_j)(2\ell+1+2m_j)}=0, \tag{3.7.5}$$

for N and each m_j positive integers with the m_i distinct: $m_i \neq m_j$, $\forall\ i \neq j$.

The product in Eq. (3.7.5) has a partial fraction decomposition of the form

$$\prod_{j=0}^{N}\frac{1}{(2\ell+1-2m_j)(2\ell+1+2m_j)}=\sum_{j=0}^{N}A_j\left[\frac{1}{(2\ell+1-2m_j)}-\frac{1}{(2\ell+1+2m_j)}\right], \tag{3.7.6}$$

where

$$A_i=\left[4m_i\prod_{j\neq i}^{N}[4(m_i^2-m_j^2)]\right]^{-1}. \tag{3.7.7}$$

Eq. (3.7.7) follows quickly from the decomposition $\dfrac{1}{(x-m)(x+m)}=\dfrac{1}{4m}\left[\dfrac{1}{x-2m}-\dfrac{1}{x+2m}\right]$. Because the sum in Eq. (3.7.5) converges absolutely, we can re-order the sums over ℓ and j, writing

$$\sum_{\ell=0}^{\infty}\prod_{j=0}^{N}\frac{1}{(2\ell+1-2m_j)(2\ell+1+2m_j)}=\sum_{j=0}^{N}A_j\sum_{\ell=0}^{\infty}\left[\frac{1}{2\ell+1-2m_j}-\frac{1}{2\ell+1+2m_j}\right]. \tag{3.7.8}$$

We now show that the sum over ℓ vanishes for any positive integer m_j. We start by noting that that the first $2m_j$ terms involving $1/(2\ell+1-2m_j)$ separately sum to zero

[9] This proof was first given in [24], and is reproduced here verbatim.

(the terms are antisymmetric about $\ell = m_j - 1/2$):

$$\begin{aligned}\sum_{\ell=0}^{2m_j-1} \frac{1}{2\ell+1-2m_j} &= \left(\sum_{\ell=0}^{m_j-1} + \sum_{\ell=m_j}^{2m_j-1}\right) \frac{1}{2\ell+1-2m_j} \\ &= \sum_{\ell=0}^{m_j-1} \frac{1}{2\ell+1-2m_j} - \sum_{\ell'=0}^{m_j-1} \frac{1}{2\ell'+1-2m_j} = 0, \qquad (3.7.9)\end{aligned}$$

where $\ell' = 2m_j - 1 - \ell$.

The remaining terms $1/(2\ell + 1 - 2m_j)$, beginning at $\ell = 2m_j$, are now identical to, and cancel, the terms $1/(2\ell + 1 + 2m_j)$, beginning at $\ell = 0$. Denoting by $\Theta(\ell - m_j)$ the step function vanishing for $\ell < m_j$, and having the value 1 for $\ell \geq m_j$, we have

$$\begin{aligned}\sum_{\ell=0}^{\infty}\left[\frac{1}{2\ell+1-2m_j} - \frac{1}{2\ell+1+2m_j}\right] &= \sum_{\ell=0}^{\infty}\left[\frac{\Theta(\ell-m_j)}{2\ell+1-2m_j} - \frac{1}{2\ell+1+2m_j}\right] \\ &= \sum_{\ell=0}^{\infty}\left[\frac{1}{2\ell+1+2m_j} - \frac{1}{2\ell+1+2m_j}\right] \\ &= 0. \ \Box\end{aligned} \qquad (3.7.10)$$

Chapter 4

The Renormalization in Electrovac

Now that we have set the foundations for regularizing particles undergoing accelerated motion, let us consider how the insights we have gained can help us understand this process for a system of interest for fundamental physics. There has been recent interest in whether self-force plays a fundamental role in enforcing cosmic censorship by preventing one from overcharging (or overspinning) a near-extreme black hole [25–28]. In this context, one would like to analyze scenarios in which gravitational and electromagnetic perturbations have comparable magnitude. The study of these scenarios introduces two new elements of this system, requiring us to analyze the singular behavior of the fields very carefully.

The first novel element is the renormalization of a pair of *coupled* divergent fields. It is not clear how this will affect the singular fields, as the metric perturbations will be caused not only by the presence of the point mass, but also by the interaction between the point charge and the background field.[1] Similarly, the singular electromagnetic field will receive contributions from not only the point charge, but also due to the modified definition of the derivative entailed by the presence of the metric perturbation.

This modification to the electromagnetic perturbation naturally leads us to the second novel element: all previous work in renormalizing a gravitational perturbation considered only renormalizing a field on a vacuum background spacetime.. So, if we were interested

[1] But not due to the stress energy of the perturbing field itself– that would be a second order correction to the self-force

in the less ambitious problem of computing the gravitational self-force on a point mass moving along a geodesic in a Riessner-Nordstrom spacetime, we would still need to extend the body of work summarized in Chapter 2 in order to account for how the distribution of the matter in the spacetime is affected by the perturbing field.

In this Chapter, we explore the results of the second paper of Linz, Friedman, and Wiseman, [29].[2] The primary result is somewhat surprising; the coupling of the fields does not play any role in the renormalization. That is, the renormalized mass is obtained by subtracting (1) the purely electromagnetic contribution from a point charge moving along an accelerated trajectory and (2) the purely gravitational contribution from a point mass moving along the same trajectory. In the context of mode-sum renormalization, this means that the required regularization parameters are sums of their purely electromagnetic and gravitational values.

Once again, we will assume that the retarded fields have already been found through some other technique, and we will instead focus on recovering the singular fields needed to renormalize these retarded fields.

4.1 The Perturbed Fields in Electrovac Spacetimes

We consider a point particle of mass $\mathtt{m}$ and charge e moving with trajectory $z(\tau)$ in a smooth electrovac spacetime, $(M, g_{\alpha\beta}, F_{\alpha\beta})$, with $F_{\alpha\beta}$ a source-free electromagnetic field. The metric $g_{\alpha\beta}$ of the background spacetime then has as its source the stress-energy tensor of $F_{\alpha\beta}$,

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} = 2\left(F_{\alpha\mu}F_{\beta}{}^{\mu} - \frac{1}{4}g_{\alpha\beta}F^{\mu\nu}F_{\mu\nu}\right), \qquad (4.1.1)$$

where $F_{\alpha\beta}$ satisfies

$$\nabla_{\beta}F^{\alpha\beta} = 0, \qquad \nabla_{[\alpha}F_{\beta\delta]} = 0. \qquad (4.1.2)$$

[2]As we reached our results, we found that Zimmerman and Poisson [30] were simultaneously studying the same systems. After a discussion with them, we were able to compare our results and determine that the approaches used were different enough to warrant separate publications. We borrow their results in section 4.2 in order to demonstrate how the qualitative results we find here are also valid for a point mass carrying a scalar charge moving in a scalarvac spacetime.

We are interested in the self-force per unit mass on the particle at linear order in $\mathtt{m}$ and e. More precisely, one could consider a family of solutions $\mathbf{g}_{\alpha\beta}(\mathtt{m}, e), \mathbf{F}_{\alpha\beta}(\mathtt{m}, e)$ whose source for nonzero $\mathtt{m}$ and e is a body of finite extent, where $e/\mathtt{m}$ has a finite limit as $\mathtt{m} \to 0$ and where the characteristic spatial length of the body is, like e, linear in $\mathtt{m}$ for small $\mathtt{m}$. At $\mathtt{m} = 0$, the spacetime is the electrovac background, and the $\mathtt{m} \to 0$ limit of the family of trajectories is given by the Lorentz force law of that background,

$$a_\alpha = \frac{e}{\mathtt{m}} F_{\alpha\beta} u^\beta, \tag{4.1.3}$$

where u^α is the particle's velocity, $a^\alpha = u^\beta \nabla_\beta u^\alpha$ is its acceleration relative to the background geometry, and ∇_α is the covariant derivative of the background metric. The self-force arises from the perturbations in the gravitational and electromagnetic fields due to the body. We denote by δQ the linear perturbation in a quantity $\mathbf{Q}(\mathtt{m}, e)$,

$$\delta Q := \mathtt{m}\, \frac{\partial}{\partial \mathtt{m}} \mathbf{Q}(m, e)\bigg|_{(\mathtt{m},e)=(0,0)} + e\, \frac{\partial}{\partial e} \mathbf{Q}(\mathtt{m}, e)\bigg|_{(\mathtt{m},e)=(0,0)} . \tag{4.1.4}$$

Then $\mathbf{Q}(\mathtt{m}, e) = Q + \delta Q + O(\mathtt{m}^2, e\mathtt{m}, e^2)$, where $Q \equiv Q(0,0)$. The perturbations $h_{\alpha\beta} = \delta g_{\alpha\beta}$ and $\delta F_{\alpha\beta}$ are the linearized gravitational and electromagnetic fields of a point particle with trajectory described by Eq. (4.1.3). In the problems that motivate this approximation, the background spacetime is nonradiative and the perturbations are the retarded fields $h^{\rm ret}_{\alpha\beta}$ and $\delta F^{\rm ret}_{\alpha\beta}$ of the particle, but the renormalization procedure is unrelated to these restrictions.

In the remainder of this Chapter, as in the previous paragraph, the symbols $g_{\alpha\beta}$ and $F_{\alpha\beta}$ will refer to the background metric and electromagnetic field. Quantities that refer to the total quantity will be written in boldface, so that $\mathbf{g}_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta}$ is the full metric and $\mathbf{F}_{\alpha\beta} = F_{\alpha\beta} + \delta F_{\alpha\beta}$ is the full electromagnetic field.

For a smooth perturbation $\mathbf{g}_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta}$, $F_{\alpha\beta} + \delta F_{\alpha\beta}$ of the geometry and electromagnetic field, the 4-velocity $\widetilde{\mathbf{u}}^\alpha$ of the perturbed trajectory, normalized with respect to the full metric satisfies

$$\mathtt{m}(g_{\alpha\beta} + h_{\alpha\beta})\widetilde{\mathbf{u}}^\gamma(\nabla_\gamma + \delta\nabla_\gamma)\widetilde{\mathbf{u}}^\beta = e(F_{\alpha\beta} + \delta F_{\alpha\beta})\widetilde{\mathbf{u}}^\beta. \tag{4.1.5}$$

Grouping terms involving the perturbed 4-velocity $\widetilde{\mathbf{u}}^\alpha$ on the left, we have

$$\mathtt{m} g_{\alpha\beta}\, \widetilde{\mathbf{u}}^\gamma \nabla_\gamma \widetilde{\mathbf{u}}^\beta - eF_{\alpha\beta}\widetilde{\mathbf{u}}^\beta \;=\; e\delta F_{\alpha\beta}\, u^\beta - \mathtt{m}\left[h_{\alpha\beta}a^\beta + \left(\nabla_\beta h_{\alpha\gamma} - \frac{1}{2}\nabla_\alpha h_{\beta\gamma}\right) u^\beta u^\gamma\right], \tag{4.1.6}$$

where we have kept only terms up to linear order in the perturbed fields. The right side plays the role of a force due to the perturbed fields, for a trajectory parameterized by proper time with respect to the full metric, $\mathbf{g}_{\alpha\beta}$. It is more common to parameterize the trajectory by proper time with respect to the background metric, $g_{\alpha\beta}$. The 4-velocity $\mathbf{u}^\alpha$ is then normalized by

$$g_{\alpha\beta}\mathbf{u}^\alpha \mathbf{u}^\beta = -1, \tag{4.1.7}$$

and we have $\mathbf{u}^\alpha = (1 - h_{\beta\gamma}u^\beta u^\gamma/2)\widetilde{\mathbf{u}}^\alpha + O(h^2)$. With this parameterization, the self-force is orthogonal to the unperturbed 4-velocity u^α. Recalling the definition of the projection operator orthogonal to u^α,

$$q^\alpha_\beta = \delta^\alpha_\beta + u^\alpha u_\beta, \tag{4.1.8}$$

the equation of motion takes the form

$$\mathtt{m} g_{\alpha\beta}\mathbf{u}^\gamma \nabla_\gamma \mathbf{u}^\beta - eF_{\alpha\beta}\mathbf{u}^\beta := f_\alpha = f^{EM}_\alpha + f^{GR}_\alpha, \tag{4.1.9}$$

where f^{EM}_α and f^{GR}_α, the contributions from the electromagnetic and metric perturbations, are given by

$$f^{EM}_\alpha \;=\; e\delta F_{\alpha\beta}u^\beta, \tag{4.1.10a}$$
$$f^{GR}_\alpha \;=\; -\mathtt{m} q^\beta_\alpha \left[(\nabla_\gamma h_{\beta\delta} - \frac{1}{2}\nabla_\beta h_{\gamma\delta})u^\gamma u^\delta + h_{\beta\gamma}a^\gamma + \frac{1}{2}h_{\gamma\delta}u^\gamma u^\delta a_\beta\right]. \tag{4.1.10b}$$

Note that in Eq. (2.5.26) the symbol f^{GR}_α denotes the expression without last two terms, the terms proportional to the background acceleration a^α. For the remainder of this Chapter, all indices will be raised and lowered by the background metric, and the perturbed trajectory will be parameterized by proper time τ with respect to the background metric.

As we stated in section 2.4, when the unperturbed motion is geodesic, the renormalized self-force at a point z of the particle's trajectory can be obtained as the $\rho \to 0$ limit of an angle average of f_α^{ret} over a sphere S_ρ of geodesic distance ρ from z [23]. Explicitly,

$$f_\alpha^R(z) = \lim_{\rho\to 0}\langle f_\alpha^{ret}\rangle_\rho = \frac{1}{4\pi}\lim_{\rho\to 0}\int_{S_\rho} d\Omega f_\alpha^{ret}, \tag{4.1.11}$$

where the components f_α^{ret} are given in Riemann normal coordinates (RNCs) centered at z. (Equivalently, the average is taken in the tangent space at z with f_α^{ret} pulled back by the exponential map.) When the trajectory is accelerated, the angle average leaves a term proportional to a_α/ρ, which can be regarded as a renormalization of the mass, $\mathtt{m}^S$. The renormalized self-force on an electromagnetic or scalar charge moving on an accelerated trajectory has the form

$$f_\alpha^R = \lim_{\rho\to 0}\left[\langle f_\alpha^{ret}\rangle_\rho - \mathtt{m}^S(\rho)a_\alpha\right], \tag{4.1.12}$$

with $\mathtt{m}^S(\rho) \propto \rho^{-1}$. For the more general situation we consider here, with electromagnetic and gravitational perturbations each contributing to the self-force, we again assume that f_α^R is given by Eq. (4.1.12).

We assume that, to linear order in the perturbed fields, the trajectory $z(\tau)$ of the particle satisfies the renormalized Lorentz-force law equation, associated with the perturbed metric $g_{\alpha\beta} + h_{\alpha\beta}$ and electromagnetic field $F_{\alpha\beta} + \delta F_{\alpha\beta}$,

$$\mathtt{m}\mathbf{g}_{\alpha\beta}\mathbf{u}^\gamma(\mathtt{m},e)\nabla_\gamma\mathbf{u}^\beta(\mathtt{m},e) - e\mathbf{F}_{\alpha\beta}\mathbf{u}^\beta(\mathtt{m},e) = f_\alpha^R + o(\mathtt{m}^2, e\mathtt{m}, e^2), \tag{4.1.13}$$

where f_α^R is obtained from the formal expression $f_\alpha^{ret} = f_\alpha^{EM,ret} + f_\alpha^{GR,ret}$ of Eq. (4.1.10) for the self-force by angle average and mass renormalization, as in Eq. (4.1.12).

We will show that the renormalization of Eq. (4.1.12) is equivalent to separate renormalization of the electromagnetic and gravitational contributions to the self-force f_α^R. It will then follow that in the mode-sum renormalization, there is no mixing of gravitational and electromagnetic parts: The renormalization is equivalent to subtracting (1) a singular expression $f_\alpha^S = f_\alpha^{EM,S} + f_\alpha^{GR,S}$, where $f_\alpha^{EM,S}$ is the purely electromagnetic contribution from a point charge moving along an accelerated trajectory (with no perturbed gravitational field); and $f_\alpha^{GR,S}$ is the the purely gravitational contribution from a point mass moving along the same trajectory that would arise if there were no perturbed electromagnetic field.

As in Chapter 2, we consider the field in a convex normal neighborhood C of the event

$z(0)$, denote by x any point of C and by ϵ the length of the unique geodesic from $z(0)$ to x (see Fig. 4). We choose $\tau = 0$ at the position of the particle where we renormalize.

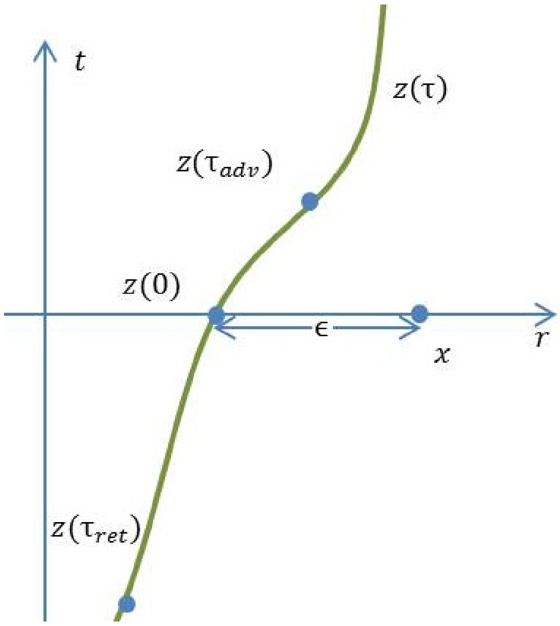

Figure 4: The particle trajectory $z(\tau)$ and a field point, x. A geodesic from $z(0)$ to x has length ϵ.

We work in a Lorenz gauge for each field. In this gauge, it is most useful to introduce the trace-reversed metric perturbation

$$\gamma_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} h^{\delta}{}_{\delta} \tag{4.1.14}$$

and a vector potential δA_α for which $\delta F_{\alpha\beta} = \nabla_\alpha \delta A_\beta - \nabla_\beta \delta A_\alpha$. These two perturbing fields satisfy the gauge conditions

$$\nabla^\beta \gamma_{\alpha\beta} = 0, \qquad \nabla^\beta \delta A_\beta = 0. \tag{4.1.15}$$

In this gauge, the perturbed Einstein equation, $\delta G_{\alpha\beta} = 8\pi \delta T_{\alpha\beta}$, has the form

$$\begin{aligned} -2\delta G_{\alpha\beta} &= \Box \gamma_{\alpha\beta} + 2\Omega_\alpha{}^\gamma{}_\beta{}^\delta \gamma_{\gamma\delta} \\ &= -16\pi \mathrm{m} \int u_\alpha u_\beta \delta^{(4)}(x, z(\tau)) d\tau - 8 \left(F_{(\alpha}{}^\delta \delta_{\beta)}{}^\gamma - \frac{1}{4} g_{\alpha\beta} F^{\gamma\delta} \right) \delta F_{\gamma\delta} \\ &\quad + \Big[4F_\alpha{}^\gamma F_\beta{}^\delta - 2F_{\alpha\epsilon} F_\beta{}^\epsilon g^{\gamma\delta} - 2g_{\alpha\beta} F^\gamma{}_\epsilon F^{\delta\epsilon} \\ &\quad + F_{\epsilon\lambda} F^{\epsilon\lambda} \left(\delta^\gamma_\alpha \delta^\delta_\beta + \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta} \right) \Big] \gamma_{\gamma\delta}, \end{aligned} \tag{4.1.16}$$

where $\Box = \nabla_\alpha \nabla^\alpha$ and

$$\Omega_{\alpha\ \beta}^{\ \gamma\ \delta} := R_{(\alpha\ \beta)}^{\ \ \gamma\ \ \delta} - R^\gamma_{(\alpha}\delta^\delta_{\beta)} - \frac{1}{2}g_{\alpha\beta}R^{\gamma\delta} + \frac{1}{2}R\delta^\gamma_{(\alpha}\delta^\delta_{\beta)}. \quad (4.1.17)$$

To make simplify the notation, we combine the last line of Eq. (4.1.16) with the term $2\Omega_{\alpha\ \beta}^{\ \gamma\ \delta}\gamma_{\gamma\delta}$ which allows us to write

$$\begin{aligned}\Box\gamma_{\alpha\beta} + 2\hat{\Omega}_{\alpha\ \beta}^{\ \gamma\ \delta}\gamma_{\gamma\delta} &= -16\pi \mathrm{m}\int u_\alpha u_\beta \delta^{(4)}(x, z(\tau))d\tau \\ &\quad -16\left(F_{(\alpha}^{\ \ [\delta}\delta_{\beta)}^{\ \ \gamma]} - \frac{1}{4}g_{\alpha\beta}F^{\gamma\delta}\right)\partial_\gamma \delta A_\delta, \end{aligned} \quad (4.1.18)$$

where

$$\begin{aligned}\hat{\Omega}_{\alpha\ \beta}^{\ \gamma\ \delta} : &= \Omega_{\alpha\ \beta}^{\ \gamma\ \delta} - 2F_{(\alpha}^{\ \ \gamma}F_{\beta)}^{\ \ \delta} + F_{(\alpha}^{\ \ \epsilon}F_{\beta)\epsilon}g^{\gamma\delta} + g_{\alpha\beta}F^\gamma_{\ \epsilon}F^{\delta\epsilon} \\ &\quad -\frac{1}{2}F_{\epsilon\gamma}F^{\epsilon\gamma}\left(\delta^\gamma_{(\alpha}\delta^\delta_{\beta)} + \frac{1}{2}g_{\alpha\beta}g^{\gamma\delta}\right). \end{aligned} \quad (4.1.19)$$

The perturbed Maxwell equation, $\delta(\nabla_\beta F^{\alpha\beta}) = 4\pi\delta j^\alpha$, is given by

$$\begin{aligned}\Box\delta A_\alpha - R^\beta_\alpha \delta A_\beta &= -4\pi e\int u_\alpha \delta^{(4)}(x, z(\tau))d\tau \\ &\quad -\nabla^\beta\left[\left(F^\gamma_{\ \beta}\delta^\delta_\alpha + F_\alpha^{\ \delta}\delta^\gamma_\beta - \frac{1}{2}g^{\gamma\delta}F_{\alpha\beta}\right)\gamma_{\gamma\delta}\right]. \end{aligned} \quad (4.1.20)$$

To find the singular behavior of the perturbed fields $\gamma_{\alpha\beta}$ and δA_α, we follow the formalism described in Chapter 2; we introduce Riemann normal coordinates (RNCs)[3] $\{x^\mu\}$ with origin at $z(0)$ and find the coordinate expansion of the perturbed fields. As in the case of particles with purely electromagnetic or gravitational interactions, the angle-average renormalization of Eq. (4.1.12) is equivalent to identifying and subtracting from f^{ret}_α a singular part f^S_α, for which the difference $f^{ret}_\alpha - f^S_\alpha$ is continuous at the position of the particle. The singular expression f^S_α is in turn obtained from Eq. (4.1.10) by replacing $\gamma_{\alpha\beta}$ and δA_α by singular parts $\gamma^S_{\alpha\beta}$ and δA^S_α of the perturbed fields. A comparison with the singular potentials found by Poisson and Zimmerman [30] using the Detweiler-Whiting singular fields shows that shows that the angle-average renormalization is again equivalent to the renormalizing using the Detweiler-Whiting prescription for the renormalized Green's functions.

[3]Since we will not be changing coordinate systems in this Chapter, we will drop the convention of 'hatting' the indices when an expression is computed in RNCs.

In order to make the greatest use of the results from Chapter 2, we decompose the field perturbations into two pieces, $\gamma^S_{\alpha\beta} = {}_I\gamma_{\alpha\beta} + {}_{II}\gamma_{\alpha\beta}$ and $\delta A^S = {}_I A_\alpha + {}_{II}A_\alpha$, satisfying

$$\Box\, {}_I\gamma_{\alpha\beta} + 2\hat{\Omega}_\alpha{}^\gamma{}_\beta{}^\delta\, {}_I\gamma_{\gamma\delta} = -16\pi\mathtt{m}\int u_\alpha u_\beta \delta^{(4)}(x, z(\tau))d\tau, \tag{4.1.21}$$

$$\Box\, {}_I A_\alpha - R^\beta_\alpha\, {}_I A_\beta = -4\pi e\int u_\alpha \delta^{(4)}(x, z(\tau))d\tau, \tag{4.1.22}$$

and

$$\Box\, {}_{II}\gamma_{\alpha\beta} + 2\hat{\Omega}_\alpha{}^\gamma{}_\beta{}^\delta\, {}_{II}\gamma_{\gamma\delta} = -16\Lambda_{\alpha\beta}{}^{\gamma\delta}\partial_\gamma \delta A_\delta, \tag{4.1.23}$$

$$\Box\, {}_{II}A_\alpha - R^\beta_\alpha\, {}_{II}A_\beta = -2\nabla^\beta\left[\Lambda^{\gamma\delta}{}_{\alpha\beta}\gamma_{\gamma\delta}\right], \tag{4.1.24}$$

where

$$\Lambda_{\alpha\beta}{}^{\gamma\delta} = F_{(\alpha}{}^{[\delta}\delta_{\beta)}{}^{\gamma]} - \frac{1}{4}g_{\alpha\beta}F^{\gamma\delta}. \tag{4.1.25}$$

At dominant order in ϵ for each of the four pieces, this is the decomposition of Eq. (4.1.4):

$${}_I\gamma_{\alpha\beta} = \mathtt{m}\,\frac{\partial}{\partial\mathtt{m}}\gamma^S_{\alpha\beta}(\mathtt{m}, e)\Big|_{(\mathtt{m},e)=(0,0)}\left[1 + O(\epsilon)\right], \tag{4.1.26a}$$

$${}_{II}\gamma_{\alpha\beta} = e\,\frac{\partial}{\partial e}\gamma^S_{\alpha\beta}(\mathtt{m}, e)\Big|_{(\mathtt{m},e)=(0,0)}\left[1 + O(\epsilon)\right], \tag{4.1.26b}$$

$${}_I A_\alpha = e\,\frac{\partial}{\partial e}\delta A^S_\alpha(\mathtt{m}, e)\Big|_{(\mathtt{m},e)=(0,0)}\left[1 + O(\epsilon)\right], \tag{4.1.26c}$$

$${}_{II}A_\alpha = \mathtt{m}\,\frac{\partial}{\partial\mathtt{m}}\delta A^S_\alpha(\mathtt{m}, e)\Big|_{(\mathtt{m},e)=(0,0)}\left[1 + O(\epsilon)\right]. \tag{4.1.26d}$$

We can quickly find the short-distance (Hadamard) expansion of the solutions to Eqs. (4.1.21) and (4.1.22), because their forms are nearly identical, respectively, to the equations governing the gravitational perturbation due to a massive particle with no charge, and to the electromagnetic perturbation due to a charged particle whose gravitational perturbation can be neglected. Eq. (4.1.22) is in fact the electromagnetic perturbation equation of a spacetime with no background electromagnetic field, but with the present background metric; with the formal expressions for the retarded and singular fields given in Eqs. (2.5.17) and (2.5.18) respectively. Eq. (4.1.21) differs from the equation governing the metric perturbation of a point mass in a vacuum spacetime only by the substitution $R_\alpha{}^\gamma{}_\beta{}^\delta \to \hat{\Omega}_\alpha{}^\gamma{}_\beta{}^\delta$. As seen in the next section, the Hadamard expansion of the field ${}_I\gamma_{\alpha\beta}$ differs only by the same substitution in Eq. (2.5.27) The RNC expansions of solutions ${}_I\gamma_{\alpha\beta}$ and ${}_IA_\alpha$ to Eqs. (4.1.21) and (4.1.22) are given by

$$\begin{aligned}\frac{1}{\mathrm{m}}\,{}_I\gamma_{\alpha\beta} &= \frac{4u_\alpha u_\beta - 8u_{(\alpha}a_{\beta)}u_\gamma x^\gamma}{\sqrt{S}} - 2u_\alpha u_\beta R_{\gamma\delta\epsilon\lambda}\frac{x^\gamma x^\epsilon x^\mu x_\mu u^\lambda u^\delta}{3S^{3/2}} + 4u_\mu u_\nu \hat{\Omega}_{(\alpha}{}^{\mu}{}_{\beta)}{}^{\nu}\sqrt{S} \\ &+\frac{4x^\mu x^\nu}{\sqrt{S}}\Bigg[(a_\alpha a_\beta + \dot{a}_{(\alpha}u_{\beta)})(q_{\mu\nu}+u_\mu u_\nu) + 2a_{(\alpha}u_{\beta)}a_\mu u_\nu \\ &-\frac{u_{(\alpha}R_{\beta)\epsilon\gamma\sigma}u^\gamma}{3}(\delta^\epsilon{}_\mu\delta^\sigma{}_\nu + u^\epsilon\delta^\sigma{}_\mu u_\nu)\Bigg] + O(\epsilon^2). \end{aligned} \tag{4.1.27}$$

and

$$\begin{aligned}\frac{1}{e}\,{}_IA_\alpha &= \frac{u_\alpha - a_\alpha u_\beta x^\beta}{\sqrt{S}} + \frac{(u_\alpha R_{\gamma\delta} - 2u^\beta R_{\alpha(\gamma\delta)\beta})}{12\sqrt{S}}\Bigg[\delta^\gamma{}_\mu\delta^\delta{}_\nu + u^\gamma u^\delta(q_{\mu\nu}+u_\mu u_\nu) \\ &+2u^\gamma\delta^\delta{}_\nu u_\mu\Bigg]x^\mu x^\nu + \frac{[2a_\alpha u_\mu a_\nu + \dot{a}_\alpha(q_{\mu\nu}+u_\mu u_\nu)]\,x^\mu x^\nu}{2\sqrt{S}} \\ &-u_\alpha R_{\beta\gamma\delta\epsilon}\frac{x^\beta x^\delta x^\lambda x_\lambda u^\gamma u^\epsilon}{6S^{3/2}} + \frac{6R_{\alpha\beta}u^\beta - u_\alpha R}{12}\sqrt{S} + O(\epsilon^2).\end{aligned} \tag{4.1.28}$$

In Eqs. (4.1.23) and (4.1.24) for ${}_{II}\gamma_{\alpha\beta}$ and ${}_{II}A_\alpha$, the left sides involve the same linear operators as those of Eqs. (4.1.21) and (4.1.22). The right sides are constructed not only from the solutions we have just obtained for ${}_I\gamma_{\alpha\beta}$ and ${}_IA_\alpha$ but also from the fields ${}_{II}\gamma_{\alpha\beta}$ and ${}_{II}A_\alpha$ themselves. We can obtain local solutions iteratively[4], noting that each solution is higher order in ϵ than its source. In particular, the leading terms in ${}_I\gamma_{\alpha\beta}$ and ${}_IA_\alpha$ proportional to $1/\sqrt{S}$ give dominant terms in ${}_{II}\gamma_{\alpha\beta}$ and ${}_{II}A_\alpha$ of subleading order, $O(\epsilon^0)$. The first iteration then uses on the right side the leading terms in ${}_I\gamma_{\alpha\beta}$ and ${}_IA_\alpha$:

$$\Box\,{}_{II}\gamma_{\alpha\beta} + O(\epsilon^{-1}) = -16e\Lambda_{\alpha\beta}{}^{\gamma\delta}u_\delta|_{x=z(0)}\partial_\gamma\left(\frac{1}{\sqrt{S_0}}\right) \tag{4.1.29}$$

$$\Box\,{}_{II}A_\alpha + O(\epsilon^{-1}) = -8\mathrm{m}\Lambda_{\gamma\delta\alpha}{}^{\beta}u^\gamma u^\delta|_{x=z(0)}\partial_\beta\left(\frac{1}{\sqrt{S_0}}\right). \tag{4.1.30}$$

[4]It is not clear at this point whether this iterative procedure yields a unique solution, and is there no reason to expect that it should. We will address this point in section 4.4, where we demonstrate its uniqueness.

Solving Eqs. (4.1.29) and (4.1.30) as RNC expansions, we find

$$
\begin{aligned}
{}_{II}\gamma_{\alpha\beta} &= -8e\frac{u_\delta \Lambda_{\alpha\beta}{}^{\gamma\delta} q_{\gamma\epsilon} x^\epsilon}{\sqrt{S_0}} + O(\epsilon) \\
&= -2\mathrm{m}\frac{x^\gamma}{\sqrt{S_0}}\left(2a_{(\alpha}\eta_{\beta)\gamma} - 2\frac{e}{\mathrm{m}}u_{(\beta}F_{\alpha)\gamma} - \eta_{\alpha\beta}a_\gamma\right) + O(\epsilon), \qquad (4.1.31) \\
{}_{II}A_\alpha &= -4\mathrm{m}\frac{u_\gamma u_\delta \Lambda^{\gamma\delta}{}_{\alpha\beta} q^\beta_\epsilon x^\epsilon}{\sqrt{S_0}} + O(\epsilon) \\
&= -\frac{\mathrm{m}}{\sqrt{S_0}}\left[F_{\alpha\beta} + \frac{\mathrm{m}}{e}(a_\alpha u_\beta - 2u_\alpha a_\beta)\right] x^\beta + O(\epsilon). \qquad (4.1.32)
\end{aligned}
$$

Here and from now on, when the symbols a_α, u^α, $q_{\alpha\beta}$ and $F_{\alpha\beta}$ appear without explicit x dependence, they denote the values of the corresponding quantities at the position $z(0)$ of the particle.

This first iteration is already enough for the principal results: The singular part of the self-force at leading and subleading order and, in particular, its contribution to the renormalized mass are unchanged by the gravitational-electromagnetic coupling. The result is due to a remarkable cancellation of the contributions to the self-force from the two mixed terms. That is, the contributions at subleading order arising from the coupling of the electromagnetic and metric perturbations are equal and opposite. To see this, we compute the self-force using Eqs. (4.1.10) in the form

$$
f^{EM}_\alpha = e(\partial_\alpha \delta A_\beta - \partial_\beta \delta A_\alpha), \qquad (4.1.33a)
$$

$$
f^{GR}_\alpha = -\mathrm{m}q^\beta_\alpha\left[\left(\nabla_\gamma\gamma_{\beta\delta} - \frac{1}{2}\nabla_\beta\gamma_{\gamma\delta} + \frac{1}{2}a_\beta\gamma_{\gamma\delta}\right)u^\gamma u^\delta - \frac{1}{4}(\nabla_\beta + a_\beta)\gamma + \gamma_{\beta\gamma}a^\gamma\right]. \qquad (4.1.33b)
$$

Substituting ${}_{II}\gamma_{\alpha\beta}$ and ${}_{II}A_\alpha$ from Eqs. (4.1.31) and (4.1.32) gives the contributions proportional to em, namely

$$
\begin{aligned}
{}_{II}f^{EM}{}_\alpha &= -e\mathrm{m}u^\beta F_{\gamma\beta}\left(\frac{\delta^\gamma_\alpha}{\sqrt{S_0}} - \frac{q_{\alpha\delta}x^\delta x^\gamma}{S_0^{3/2}}\right) + O(\epsilon^0) \\
&= -{}_{II}f^{GR}{}_\alpha. \qquad (4.1.34)
\end{aligned}
$$

Note that the angle average of each contribution,

$$
\langle {}_{II}f^{EM/GR}{}_\alpha \rangle = \mp\frac{2}{3}e\mathrm{m}F_{\alpha\beta}u^\beta\frac{1}{\sqrt{S_0}} = \mp\frac{2}{3}\frac{\mathrm{m}^2}{\sqrt{S_0}}a_\alpha, \qquad (4.1.35)
$$

is proportional to a_α and would contribute to the mass renormalization if the terms did not cancel.

The sums ${}_I\gamma_{\alpha\beta} +_{II} \gamma_{\alpha\beta}$ and ${}_IA_\alpha +_{II} A_\alpha$ are the singular fields to $O(\epsilon^0)$:

$$\gamma^S_{\alpha\beta} = 4\frac{\mathrm{m}u_\alpha u_\beta - 2\left(\mathrm{m}a_{(\alpha}u_{\beta)}u_\epsilon + eu_\delta \Lambda_{\alpha\beta}{}^{\gamma\delta} q_{\gamma\epsilon}\right) x^\epsilon}{\sqrt{S}} + O(\epsilon), \tag{4.1.36}$$

$$\delta A^S_\alpha = \frac{eu_\alpha - \left(ea_\alpha u_\epsilon + 4\mathrm{m}u_\gamma u_\delta \Lambda^{\gamma\delta}{}_{\alpha\beta} q^\beta_\epsilon\right) x^\epsilon}{\sqrt{S}} + O(\epsilon). \tag{4.1.37}$$

We will now continue the iteration to obtain an $O(\epsilon)$ contribution ${}_{II}\gamma_{\alpha\beta}$ and ${}_{II}A_\alpha$ by including on the right side of Eqs. (4.1.23) and (4.1.24) their known expansions through $O(\epsilon^0)$. We obtain in this way the $O(\epsilon)$ contribution to the singular fields $\gamma^S_{\alpha\beta}$ and δA^S_α. In principle, one could add to the iteratively obtained field a homogeneous solution to the flat-space wave equation of the form $P^{(2n)}(x)/S_0^{n-1/2}$, where $P^{(2n)}$ is a homogeneous polynomial of degree $2n$ in the coordinates $\{x^\mu\}$. We show in Section 4.4, however, that the fields $\gamma^S_{\alpha\beta}$ and $\delta A^S_{\alpha\beta}$ obtained by our iterative method are the singular fields through sub-subleading order. Substituting the expressions (4.1.36) and (4.1.37) for $\gamma_{\alpha\beta}$ and δA_α back into Eqs. (4.1.23) and (4.1.24) respectively, we have

$$\Box\, {}_{II}\gamma_{\alpha\beta} + 2\hat{\Omega}_\alpha{}^\gamma{}_\beta{}^\delta\, {}_{II}\gamma_{\gamma\delta} = -16\Lambda_{\alpha\beta}{}^{\gamma\delta}(x)\partial_\gamma \left(\frac{eu_\delta + A_{\delta\epsilon}x^\epsilon + O(\epsilon^2)}{\sqrt{S}}\right), \tag{4.1.38}$$

$$\Box\, {}_{II}A_\alpha - R^\beta_\alpha\, {}_{II}A_\beta = -\nabla^\beta \left[\Lambda^{\gamma\delta}{}_{\alpha\beta}(x)\left(\frac{8\mathrm{m}u_\gamma u_\delta + 2\gamma_{\gamma\delta\epsilon}x^\epsilon + O(\epsilon^2)}{\sqrt{S}}\right)\gamma_{\gamma\delta}\right]. \tag{4.1.39}$$

where $A_{\alpha\beta}$ and $\gamma_{\alpha\beta\gamma}$ are defined by

$$A_{\alpha\beta} := -ea_\alpha u_\beta - 4\mathrm{m}\Lambda_{\gamma\delta\alpha\epsilon}u^\gamma u^\delta q^\epsilon_\beta, \tag{4.1.40}$$

$$\gamma_{\alpha\beta\gamma} := -8\left(\mathrm{m}a_{(\alpha}u_{\beta)}u_\gamma + e\Lambda_{\alpha\beta\delta\epsilon}q^\delta_\gamma u^\epsilon\right). \tag{4.1.41}$$

The RNC expansion of $\Lambda_{\gamma\delta}{}^{\beta\alpha}(x)$ about $z(0)$ is given by

$$\Lambda_{\gamma\delta}{}^{\beta\alpha}(x) = \Lambda_{\gamma\delta}{}^{\beta\alpha}|_{x=z(0)} + \Lambda_{\gamma\delta}{}^{\beta\alpha}{}_\epsilon x^\epsilon + O(\epsilon^2), \tag{4.1.42}$$

where

$$\Lambda_{\gamma\delta}{}^{\beta\alpha}{}_\epsilon = \partial_\epsilon \Lambda_{\gamma\delta}{}^{\beta\alpha}|_{x=z(0)} = \left(\partial_\epsilon F_{(\alpha}{}^{[\delta}\delta_{\beta)}{}^{\gamma]} - \frac{1}{4}\eta_{\alpha\beta}\partial_\epsilon F^{\gamma\delta}\right)_{x=z(0)}. \tag{4.1.43}$$

Solving Eqs. (4.1.38) and (4.1.39) for ${}_{II}\gamma_{\alpha\beta}$ and ${}_{II}A_\alpha$ to $O(\epsilon)$) and adding the result to the expansions of ${}_I\gamma_{\alpha\beta}$ and ${}_IA_\alpha$, we obtain the singular fields to sub-subleading order, namely

$$\gamma^S_{\alpha\beta} = \frac{4\mathrm{m}u_\alpha u_\beta + \gamma_{\alpha\beta\epsilon}x^\epsilon}{\sqrt{S}} + \frac{4\mathrm{m}x^\gamma x^\delta\left[\left(a_\alpha a_\beta + \dot{a}_{(\alpha}u_{\beta)}\right)\left(q_{\gamma\delta}+u_\gamma u_\delta\right) + 2a_{(\alpha}u_{\beta)}a_\gamma u_\delta\right]}{\sqrt{S}}$$
$$- \frac{4\mathrm{m}}{3}\frac{u_{(\alpha}R_{\beta)\epsilon\gamma\delta}q^\epsilon_\lambda u^\gamma x^\delta x^\lambda}{\sqrt{S}} + 4\mathrm{m}u_\gamma u_\delta \hat{\Omega}_\alpha{}^\gamma{}_\beta{}^\delta\sqrt{S} + 4\Lambda_{\alpha\beta}{}^{\gamma\delta}A_{\delta\epsilon}\left(u^\epsilon u_\lambda - \delta^\epsilon_\lambda\right)$$
$$\times\left(\delta^\lambda_\gamma\sqrt{S} + \frac{q_{\gamma\mu}x^\mu x^\lambda}{\sqrt{S}}\right) + 4eu_\delta\Lambda_{\alpha\beta}{}^{\gamma\delta}{}_\epsilon\left[\left(u^\epsilon u_\lambda - \delta^\epsilon_\lambda\right)\frac{q_{\gamma\mu}x^\lambda x^\mu}{\sqrt{S}} + q^\epsilon_\gamma\sqrt{S}\right] -$$
$$2\mathrm{m}u_\alpha u_\beta R_{\gamma\delta\epsilon\lambda}\frac{x^\gamma x^\epsilon x^\mu x_\mu u^\lambda u^\delta}{3S^{3/2}} + O(\epsilon^2), \tag{4.1.44}$$

and

$$\delta A^S_\alpha = \frac{eu_\alpha + A_{\alpha\beta}x^\beta}{\sqrt{S}} + e\left(\frac{u_\alpha R_{\gamma\delta} - 2u^\beta R_{\alpha(\gamma\delta)\beta}}{12\sqrt{S}}\right)$$
$$\times\left[\delta^\gamma_\mu\delta^\delta_\nu + 2u^\gamma u_\mu\delta^\delta_\nu + u^\gamma u^\delta\left(q_{\mu\nu}+u_\mu u_\nu\right)\right]x^\mu x^\nu$$
$$+\frac{e}{2}\left[\frac{2a_\alpha a_\delta u_\gamma + \dot{a}_\alpha(q_{\gamma\delta}+u_\gamma u_\delta)}{\sqrt{S}}\right]x^\gamma x^\delta + \frac{e}{2}u^\beta R_{\alpha\beta}\sqrt{S}$$
$$+4\mathrm{m}\frac{\Lambda_{\gamma\delta\alpha\beta}u^\beta u^\gamma u^\delta u_\mu a_\nu x^\mu x^\nu}{\sqrt{S}}$$
$$+ \frac{4\mathrm{m}u_\gamma u_\delta\Lambda^{\gamma\delta}{}_{\alpha\beta\epsilon} + \Lambda^{\gamma\delta}{}_{\alpha\beta}\gamma_{\gamma\delta\epsilon}}{2}\left(u^\epsilon u_\lambda - \delta^\epsilon_\lambda\right)\left(\eta^{\lambda\beta}\sqrt{S} + \frac{q^\beta_\mu x^\lambda x^\mu}{\sqrt{S}}\right)$$
$$-eu_\alpha R_{\beta\gamma\delta\epsilon}\frac{x^\beta x^\delta x^\lambda x_\lambda u^\gamma u^\epsilon}{6S^{3/2}} + O(\epsilon^2). \tag{4.1.45}$$

In the singular fields we have just obtained, the sub-subleading terms are even functions of the coordinates x^μ. Because the expressions for the self-force in Eqs. (4.1.33) are proportional to the gradients of the potentials, they are odd in x^μ and will therefore vanish upon angle averaging. The remaining contributions to the self-force are at leading and subleading order, $O(\epsilon^{-2})$ and $O(\epsilon^{-1})$, and we find

$$f^S_\alpha = \left(e^2 - \mathrm{m}^2\right)\left[\frac{q_{\alpha\beta}x^\beta}{S_0^{3/2}} - \left[q_{\alpha\beta}a_\gamma\left(3\eta_{\epsilon\delta} - 2q_{\epsilon\delta}\right) - a_\alpha q_{\gamma\delta}\eta_{\epsilon\beta}\right]\frac{x^\gamma x^\delta x^\beta x^\epsilon}{S_0^{5/2}} - \frac{a_\alpha}{\sqrt{S_0}}\right]. \tag{4.1.46}$$

As with the uncoupled fields in Chapter 2, terms of order $O(\epsilon^0)$ can be written as a seventh order polynomial in x^μ divided by $S_0^{7/2}$, manifestly odd in the RNCs. This implies not only their angle average vanishes, but also that they do not contribute to the regularization parameters A_α and B_α in mode-sum regularization.

With the cancellation of the mixed terms in the expression for the self-force, f^S_α at subleading order is unaltered by the coupling of the electromagnetic and gravitational fields when it is written in terms of $g_{\alpha\beta}$, u_α, a_α and the RNCs. A charge e, moving with this acceleration in a geometry with this metric but with no background electromagnetic field, has f^S_α given by the part of the present f^S_α that is proportional to e^2; and a mass m, again moving on the same accelerated trajectory but with non-gravitational interactions ignored, has an f^S_α given by the terms proportional to $\mathtt{m}^2$.

In section 4.2, using the potentials obtained by Zimmerman and Poisson for a particle of scalar charge q and mass m moving in a scalarvac spacetime, we find that the analogous result holds. Again to subleading order, there is no mixed contribution to the singular expression for the self-force; f^S_α is at this order the sum of its purely gravitational and scalar terms, and the mode-sum regularization requires only parameters A_α and B_α that are each the sum of independent gravitational and scalar parameters.

4.2 Decoupling in Renormalization of a Massive Scalar Charge.

Using work on a massive scalar charge by Zimmerman and Poisson [30] (ZP), we verify here that there is no cross-term at subleading order in the singular expression for the self-force of a massive particle with scalar charge moving in a background scalarvac spacetime. The result implies that, as in the case of a point charge in an electrovac spacetime, the renormalized mass is obtained by subtracting (1) the scalar-field contribution from a point charge moving along an accelerated trajectory and (2) the purely gravitational contribution from a point mass moving along the same trajectory. In a mode-sum regularization, the regularization parameters are then sums of their purely scalar and gravitational values. This is most easily seen using a system of RNCs with origin at $z(0)$, and where we choose time slices so that the field point lies on a surface orthogonal to the world-line (so that $u_\alpha = (1,0,0,0)$ and $u_\alpha x^\alpha = 0$).

Subleading terms in f^S_α due to the coupling of fields arise from terms of order ϵ^0 in Φ^S that are proportional to $\mathtt{m}$ and from terms of order ϵ^0 in $\gamma^S_{\alpha\beta}$ that are proportional to q. We consider first the contribution to the self-force from Φ^S. From Eq. (6.19) of [30], written in terms of our RNCs with origin at $z(0)$, we have

$$\Phi^S = \frac{1}{\sqrt{S_0}}[\gamma_1 U + u_\alpha x^\alpha \gamma_2 \dot{U} + O(\epsilon^2)], \tag{4.2.1}$$

where γ_1 and γ_2 are independent of the perturbed fields, with $\gamma_1[z(0)] = \gamma_2[z(0)] = 1$. From Eq. (7.25) of [30], U and $\dot{U}$ have no terms proportional to $\mathtt{m}$, which is to say that coupling of the fields does not effect the scalar field until at least sub-subleading order, and therefore, it cannot effect $\nabla_\alpha \Phi^S$ until sub-subleading order ($O(\epsilon^0)$).

We turn next to the contribution from $\gamma^S_{\alpha\beta}$. The symbol $\hat{r}$ in ZP is $\hat{r} = u_\epsilon x^\epsilon$. Again from Eq. (6.19),

$$\gamma_S^{\alpha\beta} = \frac{1}{\rho}[\gamma_1 U^{\alpha\beta} + u_\epsilon x^\epsilon \gamma_2 \dot{U}^{\alpha\beta} + O(\epsilon^2)].$$

From Eq. (7.25), careful inspection reveals that, to relevant order, $U^{\alpha\beta}$ is identical to the term one would find for an uncharged massive particle. When one considers $\dot{U}^{\alpha\beta}$, there is a single term which arises from the coupling of the fields, namely

$$\dot{U}^{\alpha\beta}_{coupling} = -4q\dot{\Phi}u^\alpha u^\beta.$$

From Eq. (6.21), $\gamma_2 = 1 + O(\epsilon)$, and the single term arising from the coupling of the two fields in $\gamma^{\alpha\beta}$ is then

$$\gamma^{\alpha\beta}_{coupling} = -4\frac{u_\gamma x^\gamma}{\sqrt{S_0}} q\dot{\Phi}u^\alpha u^\beta.$$

The contribution of this term to the self-force at subleading order is then

$$\frac{\mathtt{m}}{4}\left[q_i^\beta(q^{\gamma\delta} + u^\gamma u^\delta) - 4q_i^\gamma u^\beta u^\delta\right]\nabla_\beta\, \gamma_{\gamma\delta}\Big|_{t=0} = \mathtt{m}q\dot{\Phi}\left[\frac{u_\beta x^\beta q_{i\gamma}x^\gamma}{S_0^{3/2}}\right]_{t=0} = 0, \tag{4.2.2}$$

using $u_i = 0$. We conclude that there is no contribution to the self-force through subleading order due to the coupling of the two fields.

4.3 Gravitational Green's Function in a Non-Vacuum Spacetime

We will make extensive use of the treatment found in [35]. The goal is to find the Green's function $G^{\alpha\beta}{}_{\gamma'\delta'}(x, x')$, where x and x' are two arbitrary points in a convex normal neighborhood C, and unprimed and primed indices are tensor indices at x and x', respectively.

When we apply this to solve for ${}_I\gamma_{\alpha\beta}$ in Eq. (4.1.27), we set $x' = z(0)$. We consider the purely gravitational Green's function, the solution to

$$\Box G^{\alpha\beta}{}_{\gamma'\delta'}(x,x') + 2\hat{\Omega}^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} G^{\mu\nu}{}_{\gamma'\delta'}(x,x') = -4\pi g^{(\alpha}{}_{\gamma'}(x,x') g^{\beta)}{}_{\delta'}(x,x')\delta^{(4)}(x,x'), \qquad (4.3.1)$$

where $g^{\alpha}{}_{\gamma'}(x,x')$ is the bivector of parallel transport, taking a vector, $v^{\gamma'}(x')$, defined at x' and parallel transporting it along the unique geodesic connecting x and x', resulting in $v^{\alpha}(x,x') = g^{\alpha}{}_{\gamma'} v^{\gamma'}(x')$.

The retarded and advanced Green's functions $G^{\alpha\beta}{}_{\gamma'\delta'\pm}(x,x')$ have the form,

$$G^{\alpha\beta}{}_{\gamma'\delta'\pm}(x,x') = U^{\alpha\beta}{}_{\gamma'\delta'}(x,x')\delta_{\pm}(\sigma) + V^{\alpha\beta}{}_{\gamma'\delta'}(x,x')\theta_{\pm}(-\sigma), \qquad (4.3.2)$$

where the distributions $\delta_{\pm}$ and $\theta_{\pm}$ are defined in Section 13 of [35], and σ is Synge's world function. Substituting Eq. (4.3.2) into the left hand side of Eq. (4.3.1), we find (with the argument (x,x') of bitensors suppressed)

$$\begin{aligned}
\Box G^{\alpha\beta}{}_{\gamma'\delta'} + 2\hat{\Omega}^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} G^{\mu\nu}{}_{\gamma'\delta'} &= -4\pi U^{\alpha\beta}{}_{\gamma'\delta'}\delta^{(4)}(x,x') \\
&\quad + \delta'_{\pm}(\sigma)\left(2U^{\alpha\beta}{}_{\gamma'\delta';\gamma}\sigma^{\gamma} + (\sigma^{\gamma}{}_{\gamma} - 4)U^{\alpha\beta}{}_{\gamma'\delta'}\right) \\
&\quad + \delta_{\pm}(\sigma)\Big(-2V^{\alpha\beta}{}_{\gamma'\delta';\gamma}\sigma^{\gamma} + (2 - \sigma^{\gamma}{}_{\gamma})V^{\alpha\beta}{}_{\gamma'\delta'} \\
&\quad + (\Box U^{\alpha\beta}{}_{\gamma'\delta'} + 2\hat{\Omega}^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} U^{\mu\nu}{}_{\gamma'\delta'})\Big) \\
&\quad + \theta_{\pm}(-\sigma)\left(\Box V^{\alpha\beta}{}_{\gamma'\delta'} + 2\hat{\Omega}^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} V^{\mu\nu}{}_{\gamma'\delta'}\right) \\
&= -4\pi g^{(\alpha}{}_{\gamma'} g^{\beta)}{}_{\delta'}\delta^{(4)}(x,x').
\end{aligned} \qquad (4.3.3)$$

In comparing this to the corresponding (unnumbered) equation in [35] (between Eq. 16.7 and 16.8), it is clear that the only difference is that the tensor $R^{\alpha}{}_{\gamma}{}^{\beta}{}_{\delta}$ is replaced here by $\hat{\Omega}^{\alpha}{}_{\gamma}{}^{\beta}{}_{\delta}$. Following the same technique used in [35], we require that the coefficients of $\delta'_{\pm}(\sigma)$, $\delta_{\pm}(\sigma)$, and $\theta_{\pm}(\sigma)$ separately vanish. We thereby find,

$$\begin{aligned}
U^{\alpha\beta}{}_{\gamma'\delta'}(x,x') &= g^{(\alpha}{}_{\gamma'}(x,x') g^{\beta)}{}_{\delta'}(x,x')\Delta^{1/2}(x,x') \\
&= g^{(\alpha}{}_{\gamma'}(x,x') g^{\beta)}{}_{\delta'}(x,x')\left(1 + \frac{1}{12}R_{\gamma'\delta'}\sigma^{\gamma'}\sigma^{\delta'} + O(\epsilon^3)\right),
\end{aligned} \qquad (4.3.4)$$

and

$$V^{\alpha\beta}{}_{\gamma'\delta'}(x',x') = \frac{\delta^{(\alpha'}{}_{\gamma'}\delta^{\beta')}{}_{\delta'}R(x')}{12} + \hat{\Omega}^{\alpha'}{}_{(\gamma'}{}^{\beta'}{}_{\delta')}(x'). \qquad (4.3.5)$$

Since $R = 0$ for electrovac, the only difference between the Hadamard expansions of a point mass in vacuum and ${}_I\gamma_{\alpha\beta}$ is in the bitensor $V^{\alpha\beta}{}_{\gamma'\delta'}$, where instead of the Riemann tensor we have $\hat{\Omega}^{\alpha'}{}_{(\gamma'}{}^{\beta'}{}_{\delta')}$.

4.4 The Iterative Method

We show that iterative solutions (4.1.44) and (4.1.45) obtained in Sect. 4.1 are the near-field expansion of the singular electromagnetic potential δA^{ret}_α and trace-reversed metric perturbation $\gamma^{ret}_{\alpha\beta}$. To do so, we use general features of the Hadamard expansion for the singular fields to constrain the form of the expansion; given these constraints, we show the iterative solution is unique. It is helpful to use RNCs $\{t, x^i\}$ for which the t =constant surface is orthogonal to u^α. We write $r := \sqrt{\delta_{ij}x^i x^j} = \sqrt{q_{\alpha\beta}x^\alpha x^\beta} = \sqrt{S_0}$.

We begin with the Detweiler-Whiting form of the singular fields, with RNC components

$$\delta A^S_\alpha(x) = \frac{1}{2}\left(\frac{\mathtt{U}_\alpha}{\dot{\sigma}}\bigg|_{ret} + \frac{\mathtt{U}_\alpha}{\dot{\sigma}}\bigg|_{adv}\right) - \int_{\tau_{ret}}^{\tau_{adv}} \mathtt{V}_\alpha(x, z(\tau))d\tau, \tag{4.4.1a}$$

$$\gamma^S_{\alpha\beta}(x) = \frac{1}{2}\left(\frac{\mathtt{U}_{\alpha\beta}}{\dot{\sigma}}\bigg|_{ret} + \frac{\mathtt{U}_{\alpha\beta}}{\dot{\sigma}}\bigg|_{adv}\right) - \int_{\tau_{ret}}^{\tau_{adv}} \mathtt{V}_{\alpha\beta}(x, z(\tau))d\tau. \tag{4.4.1b}$$

Here $\mathtt{U}_\alpha(x), \mathtt{U}_{\alpha\beta}(x), \mathtt{V}_\alpha(x)$ and $\mathtt{V}_{\alpha\beta}(x)$ are smooth tensors defined in the convex normal neighborhood C of $z(0)$, with $\mathtt{U}_\alpha := \mathtt{U}_\alpha[z(0)] = eu_\alpha,\ \mathtt{U}_{\alpha\beta} := \mathtt{U}_{\alpha\beta}[z(0)] = 4mu_\alpha u_\beta$.

The coincidence values, $\mathtt{V}_\alpha := \mathtt{V}_\alpha[z(0), z(0)]$ and $\mathtt{V}_{\alpha\beta} := \mathtt{V}_{\alpha\beta}[z(0), z(0)]$, determine the values of the integrands in Eqs. (4.4.1) at sub-subleading order, $O(\epsilon)$:

$$\int_{\tau_{ret}}^{\tau_{adv}} \mathtt{V}_\alpha(x, z(\tau))d\tau = (\tau_{adv} - \tau_{ret})\mathtt{V}_\alpha = 2r\,\mathtt{V}_\alpha + O(\epsilon^2), \tag{4.4.2a}$$

$$\int_{\tau_{ret}}^{\tau_{adv}} \mathtt{V}_{\alpha\beta}(x, z(\tau))d\tau = 2r\,\mathtt{V}_{\alpha\beta} + O(\epsilon^2). \tag{4.4.2b}$$

The iteration finds these terms and the expansion of the terms involving $\mathtt{U}_\alpha$ and $\mathtt{U}_{\alpha\beta}$. In the terms involving $\mathtt{U}_\alpha$ and $\mathtt{U}_{\alpha\beta}$, $1/\dot{\sigma}_{ret/adv}$ depends only on the background spacetime and the trajectory and is the same for each field. Its expansion is given by

$$\frac{1}{\dot{\sigma}_{ret/adv}} = \frac{1}{r}\left[1 - \frac{1}{2}\left(1 - \frac{t^2}{r^2}\right)a_\alpha x^\alpha + O(\epsilon^2)\right] \tag{4.4.3}$$

Because $1/\dot{\sigma}_{ret} = 1/\dot{\sigma}_{adv} + O(\epsilon)$ and $\mathtt{U}^{ret}_\alpha(x) = \mathtt{U}^{adv}_\alpha(x) + O(\epsilon)$, only the combination $\mathtt{U}^S_\alpha(x) := \frac{1}{2}[\mathtt{U}^{ret}_\alpha(x) + \mathtt{U}^{adv}_\alpha(x)]$ appears in the expansion at sub-subleading order, $O(\epsilon)$. Writing

$$\mathtt{U}^S_\alpha(x) = \mathtt{U}_\alpha + x^\gamma \partial_\gamma \mathtt{U}^S_\alpha + \frac{1}{2} x^\gamma x^\delta \partial_\gamma \partial_\delta \mathtt{U}^S_\alpha + O(\epsilon^2),$$

we have

$$\begin{aligned}\delta A^S_\alpha(x) &= \frac{1}{r}\Bigg[\mathtt{U}_\alpha + x^\gamma \partial_\gamma \mathtt{U}^S_\alpha - \frac{1}{2}\left(1 - \frac{t^2}{r^2}\right)\mathtt{U}_\alpha a_\beta x^\beta \\ &\quad + \frac{1}{2} x^\gamma x^\delta \partial_\gamma \partial_\delta \mathtt{U}^S_\alpha - \frac{1}{2}\left(1 - \frac{t^2}{r^2}\right) a_\beta \partial_\gamma \mathtt{U}^S_\alpha x^\beta x^\gamma \\ &\quad + \text{terms independent of } \mathtt{U}^S_\alpha + O(\epsilon^2)\Bigg] - r\,\mathtt{V}_\alpha. \end{aligned} \tag{4.4.4}$$

We begin by showing uniqueness at subleading order of the solution ${}_{II}A_\alpha$ to Eq. (4.1.30). To the solution given in Eq. (4.1.32), one can add any f_α satisfying $\Box f_\alpha = 0$. At subleading order, however, the only term involving $\mathtt{U}^S_\alpha$ is $\frac{1}{r} x^\gamma \partial_\gamma \mathtt{U}^S_\alpha$, linear in the coordinates. At subleading order, f_α must then be linear in the coordinates, with each component having the form $a_\alpha x^\alpha = a_t \frac{t}{r} + a_i \frac{x^i}{r}$, a sum of monopole and dipole parts. Then $\Box(a_\alpha x^\alpha) = 0$ implies $a_\alpha = 0$, whence $f_\alpha = 0$, and the solution (4.1.32) is unique at subleading order.

The solution at subleading order is now used to obtain a solution at sub-subleading order, $O(\epsilon)$. Because $\partial_\gamma U_\alpha$ is now fixed, the only ambiguity in the solution allowed by the Hadamard form (4.4.4) is in the terms $\dfrac{1}{2r} x^\gamma x^\delta \partial_\gamma \partial_\delta \mathtt{U}^{sing}_\alpha$ and $-r\mathtt{V}_\alpha$: That is, the solution δA^S_α of Eq. (4.1.45) is unique at subleading order up to adding a solution to $\Box f_\alpha = 0$ for which each component is of the form $a_{\alpha\beta} x^\alpha x^\beta / r$. The spatial part $a_{ij} x^i x^j$ can be decomposed into monopole and quadrupole parts by writing $a_{ij} = \frac{1}{3}\delta_{ij} a^k_k + a^{STF}_{ij}$, where a^{STF}_{ij} is symmetric and tracefree. Then $a_{\alpha\beta} x^\alpha x^\beta / r$ is a sum of monople, dipole and quadrupole parts, namely

$$a_{\alpha\beta} x^\alpha x^\beta = \left(a_{tt}\frac{t^2}{r} + \frac{1}{3} a^k_k r\right) + 2a_{ti}\frac{t x^i}{r} + a^{STF}_{ij}\frac{x^i x^j}{r}. \tag{4.4.5}$$

Again $\Box(a_{\alpha\beta} x^\alpha x^\beta) = 0$ only if the D'Alembertian of each of these parts separately vanishes. We immediately conclude that the coefficients of the dipole and quadrupole parts vanish: $a_{ti} = 0 a^{STF}_{ij}$. For the monopole term, we have

$$\Box\left(a_{tt}\frac{t^2}{r} + \frac{1}{3} a^k_k r\right) = -4\pi a_{tt} t^2 \delta^3(x) + (2a_{tt} + \frac{2}{3} a^k_k)\frac{1}{r}, \tag{4.4.6}$$

vanishing only if $a_{tt} = 0 = a^k_k$. Thus $f_\alpha = 0$, and the solution δA^{sing}_α is unique through subleading order.

The proof of uniqueness for $\gamma^S_{\alpha\beta}$ is essentially identical, and is obtained by replacing ${}_{II}A_\alpha$, δA^S_α, $\mathtt{U}_\alpha$, and $V_{\alpha\beta}u^\beta$ by ${}_{II}\gamma_{\alpha\beta}$, $\gamma^S_{\alpha\beta}$, $\mathtt{U}_{\alpha\beta}$, and $V_{\alpha\beta\gamma\delta}u^\gamma u^\delta$ respectively.

4.5 Discussion

We have demonstrated how to renormalize in electrovac based on the angle-average and mass renormalization ansatz given in Eq. (4.1.12),

$$f^R_\alpha = \lim_{\rho\to 0}\left[\langle f^{ret}_\alpha\rangle_\rho - \mathtt{m}^S(\rho)a_\alpha\right].$$

By splitting our perturbations into two pieces, we were able to identify familiar solutions (${}_I\gamma_{\alpha\beta}$ and ${}_I\delta A_\alpha$) which dominate for low ϵ. Using these fields we solved iteratively for the new solutions (${}_{II}\gamma_{\alpha\beta}$ and ${}_{II}\delta A_\alpha$) arising due to the coupling of the gravitational and electromagnetic fields. Furthermore, we were able to demonstrate that this method will guarantee that we recover the true DW singular fields.

Due to a surprising cancellation, we find that the coupling of the fields does not effect the renormalized mass, so that the values of the regularization parameters A_α and B_α are the sums of the values for the purely gravitational and purely electromagnetic contributions to the regularization parameters of an accelerated particle with either mass $\mathtt{m}$ or with charge e. [5] Using the results of Zimmerman and Poisson [30] we demonstrated similar behavior for a massive scalar particle moving through scalarvac.

One thing that is important to note is that our renormalization of coupled fields has not yet been rigorously justified by matched asymptotic expansions. Between our work and that of Zimmerman and Poisson we have used two different approaches and recovered the same renormalization procedure. Furthermore, Zimmerman, in a separate work[6] used effective field theory to also recover this same result. The agreement of these very different approaches is a compelling argument for their validity.

[5]The higher order regularization parameters $D^{(2j)}_\alpha$ presumably would involve terms arising form the mixing of the fields, but they multiply vanishing sums.

[6]in preparation

Even without a rigorous justification for our results, most of our results still *must hold*. Since the self-force experienced by the particle must be finite, the leading and subleading terms in the expression must be correct. In turn, this means that the $A_\alpha = A_\alpha^{GR} + A_\alpha^{EM}$, $B_\alpha = B_\alpha^{GR} + B_\alpha^{EM}$, and that $\mathtt{m}^S = \mathtt{m}^S_{GR} + \mathtt{m}^S_{EM}$, since these terms are required to make the self-force finite. What we cannot say for certain is that D_α term vanishes.

Chapter 5

Scalar Self-force for Accelerated Trajectories in Schwarzscihld

Up to this point, we have focused purely on renormalization, developing formal expressions for the equations of motion in terms of a local expression added to a tail term, and using these to derive the regularization parameters necessary for mode-sum renormalization. While this work is fundamental to understanding BHP theory and renormalization in general, we have always assumed that the expressions for the retarded fields were known but have not yet actually computed a self-force.

In this Chapter and the following, we will compute the scalar self-force for a point source moving along a non-Keplerian circular orbit in Schwarzschild spacetime. We will use the formulation from Mano, Suzuki, and Takasugi [2] to generate analytic solutions to the field equations (henceforth we will refer to their method as simply MST). Using these, along with a useful mathematical insight from Hikida et al. [4] [5], we will compute the first order scalar self-force in a perturbative, post-Newtonian-like manner. With the aid of the computer algebra program *Mathematica*, we extend this analytic solution to several orders.

After Pound et al. [49] derived the renormalization in a radiation gauge, interest in applying this MST technique to the study of the gravitational self-force grew[1]. This work

[1]Keidl et al. [43] and Shah et al. [50] were already working in a radiation gauge. The work of Pound et al. [49] provided a rigorous explanation of the renormalization

has been further expanded in Merlin and Shah [51] and even more recently in Shah and Pound [49].

Recently, Shah, Whiting, and Friedman [3] used the MST technique to generate very high-order Post-Newtonian correction terms, including many that had not been recovered previously. Even more recently, Shah and Pound [38] used these techniques to compute coefficients up to 20 pN order for the spin procession and tidal invariants in Schwarzschild.

Since these techniques have been applied to astrophysically relevant systems, it would appear odd at first to continue studying an accelerated scalar charge. Why study the scalar self-force when the techniques have already been developed for use in the gravitational system? As we will discuss in greater detail in Chapter 6, the method suggest by Hikida leads to many apparent contradictions, and so we choose to study these in their simplest form so that we can isolate these apparent contradictions from difficulties associated with fields of higher spin.

In Chapter 1, we discussed some of the abstract reasons to study accelerated orbits, the chief one being to open up a wider range of comparisons. For example, assume that we have computed the self-force for a particle traveling along a circular motion in Schwarzschild, using Kepler's law to rewrite the mass of the black hole in terms of the velocity of the particle. We will recover a complicated answer, and it would be very useful to make comparisons between this result and other simpler, well known results.

If we tried taking the static limit of an solution computed under the assumption of geodesic motion (to zeroth order), we would annihilate every term in our expressions; for geodesic motion if $v \to 0$ then $M \to 0$. Similarly, in comparing the damping force on the particle it would be reassuring to see if the expression we find has a sane flat-spacetime limit, before using the expression to model more complicated physics. Here too, we encounter the same problem. Therefore, if we attempt to take either of these limits, the field reduces to that of a static point source in flat spacetime.

By considering accelerated orbits, the expressions we obtain will necessarily become much more complicated than the would be for geodesic orbits, since, we cannot combine terms of the form $(M/r)^4$ with v^8. While this is certainly detrimental in many respects[2],

[2] for instance typesetting the equations themselves.

it could prove beneficial in others. For example, Galley has pioneered the efforts to apply effective field theory to the self-force problem (see for instance [52]), and even though $(M/r)^4$ and v^8 are of the same order, they are associated with different Feynman diagrams.

Finally, by considering accelerated circular orbits, we will recover the same expressions that we will need for considering elliptic geodesics. We can develop the machinery of MST and explore the implications of the insights from Hikida et al. [4, 5] in isolation from the additional complexity when we can no longer simply replace the Fourier frequency ω by the term $m\Omega$. In this way, accelerated circular orbits are a sort of stepping-stone towards elliptic orbits.

This Chapter's layout is as follows: First we will discuss the MST [2] formalism used to generate the solutions to the differential equations in section 5.1. In section 5.2 we discuss the methods for generating pN expansions using this method. In section 5.3, we discuss the insights from Hikida et al. [4, 5] on how to separate the Green's functions to ease the regularization procedure in section . In section 5.4, we will discuss the methods used to actually solve for the Green's functions and the forces. Finally, we will finish up this Chapter with a discussion of the damping force in section 5.5, leaving the conservative self-force to the next Chapter.

5.1 The Teukolsky Equation and the MST Formalism

In order to solve for the retarded fields in a black-hole spacetime, we solve the Teukolsky equation by writing its Fourier-harmonic decomposition,

$$\psi = \int d\omega e^{-i\omega t} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} R_{\omega\ell m}(r) S_{\ell m\omega}(\theta, \phi), \tag{5.1.1}$$

where the $S_{\ell m}(\theta, \phi)$ are the spheroidal harmonics, and the coordinates (t, r, θ, ϕ) are the standard Boyer-Lindquist coordinates. Using this decomposition, the radial Teukolsky equation can be written as

$$\left[\Delta\partial_r^2 + 2(r-M)(s+1)\partial_r + \frac{k^2 - 2is(r-M)k}{\Delta} - 4is\omega r - \lambda\right] R_{\omega\ell m} = 0, \tag{5.1.2}$$

where s is the spin of the field, a is the spin of the hole, $\Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-)$, $r_\pm = M \pm \sqrt{M^2 - a^2}$, $k = (r^2 + a^2)\omega - aM$, $\lambda = E - s(s+1) - 2Ma\omega + a^2\omega^2$, and E is the eigenvalue from the spin-weighted spheroidal harmonics.

To solve this equation, we define the following new variables:

$$\begin{aligned} &z := \omega r; \quad && z_\pm := \omega r_\pm; \quad && \epsilon := 2M\omega; \quad && \tau := \frac{\epsilon - mq}{\kappa} \\ &\kappa := \sqrt{1-q^2}; \quad && q := \tfrac{a}{M}; \quad && x := \frac{z_+ - z}{\epsilon\kappa}; \quad && \hat{z} = z - z_-. \end{aligned} \tag{5.1.3}$$

Following Sasaki et al. [53] we write

$$R_{\omega \ell m} = \hat{z}^{-1-s}\left(1 - \frac{\epsilon\kappa}{\hat{z}}\right)^{-s-i(\epsilon+\tau)/2} \phi(\hat{z}). \tag{5.1.4}$$

With this substitution we can write the radial Teukolsky equation as

$$\begin{aligned} \hat{z}^2\phi'' \;&+\; \left[\hat{z}^2 + (2\epsilon + 2is)\hat{z} - \lambda - s(s+1)\right]\phi = \\ &\epsilon\kappa\left[\hat{z}(\phi'' + \phi) + (s - 1 + i\tau + i\epsilon)\phi'\right] \\ &+ \left(-\frac{\epsilon[\kappa - i(\epsilon - mq)](s - 1 + i\epsilon)}{\hat{z}} + (\epsilon mq + (\kappa - 2)\epsilon^2 + i\kappa(\epsilon s))\right)\phi. \end{aligned} \tag{5.1.5}$$

The left hand side of Eq. (5.1.5) has the form of the Coulomb wave equation. The right hand side is of $O(\epsilon)$ so as $\epsilon \to 0$, $\phi(\hat{z})$ approaches the Coulomb wave function. For the case when $\epsilon \neq 0$, we introduce a quantity ν, called the renormalized angular momentum. We then add the quantity $(\lambda + s(s+1) - \nu(\nu+1))$ to both sides of Eq. (5.1.5), finding

$$\begin{aligned} \hat{z}^2\phi'' \;&+\; [\hat{z}^2 + 2(\epsilon + is)\hat{z} - \nu(\nu+1)]\phi = \\ &\epsilon\kappa\left[\hat{z}(\phi'' + \phi) + (s - 1 + i\tau + i\epsilon)\phi'\right] + \left(-\frac{\epsilon(\kappa - i(\epsilon - mq))(s - 1 + i\epsilon)}{\hat{z}}\right. \\ &\left. -\nu(\nu+1) + \lambda + s(s+1) - 2\epsilon^2 + \epsilon mq + \kappa\epsilon(\epsilon + is)\right)\phi \end{aligned} \tag{5.1.6}$$

Now, we will specialize to scalar fields ($s = 0$) and Schwarzschild spacetime ($a = 0$, $q = 0$, $\kappa = 1$, $\tau = \epsilon$), with line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{5.1.7}$$

In Schwarzschild spacetime, $\lambda = \ell(\ell+1)$, which lets us write

$$\phi_c^\nu = (2z)^\nu \Phi^\nu = (2z)^\nu \left(1 - \frac{\epsilon}{z}\right)^\nu \sum_{n=-\infty}^{\infty} [2i(z-\epsilon)]^n a_n^\nu \frac{(\nu+1+i\epsilon)_n}{(2\nu+2)_{2n}} \times F_{n,\nu}(z,\epsilon), \quad (5.1.8)$$

where

$$F_{n,\nu}(z,\epsilon) = e^{-i(z-\epsilon)} {}_1F_1(n+\nu+1+i\epsilon; 2n+2\nu+2; 2i(z-\epsilon)), \quad (5.1.9)$$

where ${}_1F_1(a;b;x)$ is a confluent hypergeometric function. We use the Pochammer symbol,

$$(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)}. \quad (5.1.10)$$

The a_n^ν in Eq. (5.1.8) are determined by inserting the solution ϕ_c^ν in to Eq. (5.1.6) and making use of the three term-recurrence relations for the confluent hypergeometrics to generate a three term recurrence relation for the coefficients a_n^ν:

$$\alpha_n^\nu a_{n+1}^\nu + \beta_n^\nu a_n^\nu + \gamma_n^\nu a_{n-1}^\nu = 0, \quad (5.1.11)$$

where

$$\alpha_n^\nu = i\epsilon \frac{(n+\nu+1+i\epsilon)^2(n+\nu+1-i\epsilon)}{(n+\nu+1)(2n+2\nu+3)}, \quad (5.1.12)$$

$$\beta_n^\nu = -\ell(\ell+1) + (n+\nu)(n+\nu+1) + 2\epsilon^2 + \frac{\epsilon^4}{(n+\nu)(n+\nu+1)}, \quad (5.1.13)$$

$$\gamma_n^\nu = -i\epsilon \frac{(n+\nu+i\epsilon)(n+\nu-i\epsilon)^2}{(n+\nu)(2n+2\nu-1)} \quad (5.1.14)$$

We will normalize our answers by pulling out an overall normalization term and asserting that $a_0^\nu = 1$. The three term recurrence relations are closely related to continued fractions. We define the "right mover" and "left mover" respectively,

$$R_n := \frac{a_n^\nu}{a_{n-1}^\nu} = \frac{-\gamma_n^\nu}{\beta_n^\nu + \alpha_n^\nu R_{n+1}}$$
$$L_n := \frac{a_n^\nu}{a_{n+1}^\nu} = \frac{-\alpha_n^\nu}{\beta_n^\nu + \gamma_n^\nu L_{n-1}}. \quad (5.1.15)$$

Thus, it is possible to generate the a_n^ν by successive applications of the right mover for $n > 0$ and the left mover for $n < 0$. At this point, it is useful to stop and consider two different issues with convergence. First we need to know if the solution generated by using the left mover will converge to the same answer found by using the right mover. Second, once we have these solutions, does the infinite sum over the coulomb wave functions converge?

To answer these, we will follow the discussion of a similar relation by Koranda and Allen [54]. In general, a three term recursion relation will have two independent solutions. These solutions are called *minimal* solutions if, as $|n| \to \infty$, the $a_n^\nu \to 0$. A solution which is not minimal is called a *dominant* solution. While three term recursion relationships will have two linearly independent solutions, there is no requirement that either of them are minimal solutions.

If we consider it from the point of view of the continued fractions, we can utilize Pincherle's Theorem [55], which tells us that R_n (L_n) converges for $n \geq 1$ $(n \leq -1)$ if and only if the recurrence relation has a minimal solution for $n \geq 1$ $(n \leq -1)$. Furthermore, if the right or left movers converge, then they converge to the minimal solution.

Now, imagine that we have such a minimal solution and can therefore find the a_n^ν, we can still have an unsatisfactory solution. This can happen when the right and left movers converge to *different* minimal solutions. In short, this happens because while we can generate the a_n^ν for positive n with the right mover, and the a_n^ν for negative n using the left mover, we have not related a_{-1}^ν to a_1^ν, so the negative n and positive n solutions have no way of "knowing" anything about the other solution. Fortunately, we have yet to specify the renormalized angular momentum, ν, and therefore we can use Eq. (5.1.11) for $n = 0$ to calculate ν, which will "inform" the positive n solutions of the negative n solutions and vice-versa.

Now that we have discussed the theory of these solutions, we need to make sure that our solutions actually do converge. To solve for the a_n^ν we treat ϵ as a small parameter $(\epsilon << 1)$ and expand each a_n^ν as a series in ϵ. Then we solve for the terms using the recurrence relationship, normalizing such that $a_0^\nu = 1$.

If we assume that $\nu = \ell + O(\epsilon^2)$, then by inspecting Eqs. (5.1.12-5.1.15) we can state that for all positive n, $R_n \propto O(\epsilon)$, meaning that the $a_n^\nu \propto \epsilon^n$. Because ϵ is a small parameter, $a_n^\nu \to 0$ as $n \to \infty$.

When we examine the a_n^ν for $n < 0$, we notice that the coefficients given by Eqs. (5.1.12-5.1.14) have special cases for $n = -\ell$, $n = -\ell - 1$, and $n = -2\ell - 1$. We can make the following statements (Eq. (5.3) of [19]),

$$\begin{aligned} L_{-\ell-1} &= O(\epsilon^2), \\ L_{-2\ell-1} &= O(\epsilon^{-1}), \\ L_n &= O(\epsilon), \qquad n \neq -\ell-1, n \neq -2\ell-1, \end{aligned} \tag{5.1.16}$$

which tells us that as $n \to -\infty$, $a_n^\nu \to \epsilon^\infty$, and therefore the solution for $n < 0$ is also a minimal solution.[3]

Notice that the original differential equation is symmetric under $\ell \leftrightarrow -\ell-1$ (which also means under $\nu \leftrightarrow -\nu-1$), which implies that we can also get a solution by making this substitution. This gives us a second, linearly independent solution to the field equations, where

$$a_n^{-\nu-1} = (-1)^n a_{-n}^\nu. \tag{5.1.17}$$

Thus, we will call our two linearly independent solutions to the field equations $\phi_c^\nu = (2z)^\nu \Phi^\nu$ and $\phi_c^{-\nu-1} = (2z)^{-\nu-1}\Phi^{-\nu-1}$.[4] Now that we have discussed the general process for generating our solutions, there are a few practical elements that we need to discuss in order to actually evaluate our solutions.

5.1.1 Boundary Conditions

We will introduce the new variable $\zeta := z - \epsilon = \omega(r - 2M)$ to ease notation in the following discussion. In terms of ζ, the radial solutions from Eq. (5.1.8) take the form.

$$\begin{aligned} \phi_c^\nu &= e^{-i\zeta}(2\zeta)^\nu \sum_{n=-\infty}^{\infty} i^n a_n^\nu (2\zeta)^n \frac{(\nu+1+i\epsilon)_n}{(2\nu+2)_{2n}} \\ &\quad \times {}_1F_1(n+\nu+1+i\epsilon, 2n+2\nu+2, 2i\zeta)\ . \end{aligned} \tag{5.1.18}$$

Careful comparison with Eqs. (3.1-3.4) in MST [2] reveals that this definition amounts to a change in the renormalization, namely,

$$\phi_c^\nu = \frac{\Gamma(2\nu+2)}{\Gamma(\nu+1+i\epsilon)} R_{c\,(\mathrm{LFE})}^\nu\ , \tag{5.1.19}$$

[3]For $s \neq 0$, the first inequality reads $L_{-\ell-1} = O(1)$.

[4]The subscript c here is used to emphasize that these solutions will be expanded in terms of the unnormalized Coulomb wave functions (see Eq. (5.1.18) below).

where $R^{\nu}_{c(\mathrm{LFE})}$ are the MST solutions.[5] The change in normalization is an improvement– it ensures that, aside from leading factor of $(\sigma_\omega)^\nu = (\mathrm{sign}(\omega))^\nu$, ϕ^ν_c is real. Furthermore, ϕ^ν_c now is free of any Γ functions, a vitally important property in terms of the usefulness of the techniques of Hikida et al. [4]. Using Eq. (6.7.6) from [56],

$$ {}_1F_1(a,c,x) = \frac{\Gamma(c)}{\Gamma(c-a)} e^{i\sigma_\omega a\pi}\psi(a,c,x) + \frac{\Gamma(c)}{\Gamma(a)} e^{i\sigma_\omega(a-c)\pi} e^{x}\psi(c-a,c,-x), \tag{5.1.20} $$

where $\sigma_\omega = \mathrm{sign}(\mathrm{Im}(x)) = \mathrm{sign}(\mathrm{Im}(2i\zeta)) = \mathrm{sign}(\omega)$, and Eq. (33) from [57]

$$ \frac{(\nu+1+i\epsilon)_n}{(\nu+1-i\epsilon)_n} a^\nu_n = (-1)^n (\bar{a}^\nu_n)\ , \tag{5.1.21} $$

we have

$$ \begin{aligned} \phi^\nu_c &= (\sigma_\omega)^\nu \Bigg\{ |2\zeta|^\nu e^{i\zeta} e^{i\pi\sigma_\omega(\nu+1)} \frac{e^{-\pi|\epsilon|}\Gamma(2\nu+2)}{\Gamma(\nu+1-i\epsilon)} \\ &\quad \times \sum_{n=-\infty}^{\infty} a^{\nu *}_n [2i\zeta]^n \Psi(n+\nu+1+i\epsilon, 2n+2\nu+1, 2i\zeta) \\ &\quad + |2\zeta|^\nu e^{-i\zeta} e^{-i\pi\sigma_\omega(\nu+1)} \frac{e^{-\pi|\epsilon|}\Gamma(2\nu+2)}{\Gamma(\nu+1-i\epsilon)} \\ &\quad \times \sum_{n=-\infty}^{\infty} a^\nu_n [-2i\zeta]^n \Psi(n+\nu+1-i\epsilon, 2n+2\nu+1, -2i\zeta) \Bigg\} \\ &= (\sigma_\omega)^\nu \left(\phi^\nu_{c,(\mathrm{in\ at\ }\infty)} + \phi^\nu_{c,(\mathrm{out\ at\ }\infty)} \right) . \end{aligned} \tag{5.1.22} $$

If we ignore the leading factor $(\sigma_\omega)^\nu$, which is complex when the frequency is negative, we see that the two terms in Eq. (5.2.1) are complex conjugates, meaning that ϕ^ν_c is real (up to the aforesaid leading factor) and is in fact real term-by-term in the summations.

Now, we will demonstrate that the final form in Eq. (5.2.1) shows that ϕ^ν_c can be written as sum of ingoing and outgoing solutions at infinity. It is also clear that from this Eq. (5.2.1) that

$$ \phi^\nu_{c,(\mathrm{in\ at\ }\infty)}(\omega \to -\omega) = \phi^\nu_{c,(\mathrm{out\ at\ }\infty)}. \tag{5.1.23} $$

By making the switch $\nu \to -\nu-1$, $n \to -n$ and using $a^{-\nu-1}_{-n} = a^\nu_n$ (Eq. (2.17) in [2]), we have

$$ \phi^{-\nu-1}_c = (\sigma_\omega)^{-\nu-1}(A\,\phi^\nu_{c,\mathrm{in}} + A^*\,\phi^\nu_{c,\mathrm{out}}), \tag{5.1.24} $$

[5] We have added the subscript (LFE) to emphasize that these solutions are the ones given in [2] (the Low Frequency Expansion paper), as opposed to other slightly different functions (*e.g.* different variables, different normalizations, *etc.*) that are given in other papers, yet use exactly the same notation.

where

$$A = \frac{\sin \pi(\nu + i\epsilon)\ |\Gamma(\nu + 1 \pm i\epsilon)|^2}{\sin 2\pi\nu\, \Gamma(2\nu + 1)\Gamma(2\nu + 2)} e^{-i\pi\sigma_\omega(\nu+1/2)}. \tag{5.1.25}$$

The Asymptotic Behavior of ϕ_c^ν and $\phi_c^{-\nu-1}$

Using the asymptotic relation $\Psi(a, b, z) \xrightarrow{|z|\to\infty} z^{-a}(1+O(1/z))$, [Abramowitz and Stegun [58], Eq. (13.1.8)] we have

$$\phi^\nu_{c,(\text{in at } \infty)} \xrightarrow{r\to\infty} \frac{1}{2|\zeta|} \frac{e^{-\pi|\epsilon|/2}\,\Gamma(2\nu + 2)}{|\Gamma(\nu + 1 - i\epsilon)|} \rho_a e^{-i(\zeta+\epsilon \ln 2|\zeta|+\phi_a-\phi_\Gamma/2-\pi\sigma_\omega(\nu+1)/2)} \tag{5.1.26}$$

where the various quantities are defined by

$$\sum_{n=-\infty}^{\infty} a^\nu_n \equiv \rho_a e^{i\phi_a} \tag{5.1.27}$$

$$\frac{\Gamma(\nu + 1 + i\epsilon)}{\Gamma(\nu + 1 - i\epsilon)} \equiv e^{i\phi_\Gamma} . \tag{5.1.28}$$

We can construct the out-going solution at infinity by

$$\phi^\nu_{c,(\text{out at } \infty)} = \phi^\nu_{c,(\text{in at } \infty)}(\omega \to -\omega) = \bar{\phi}^\nu_{c,(\text{in at } \infty)} \tag{5.1.29}$$

$$\xrightarrow{r\to\infty} \frac{1}{2|\omega|r} \frac{e^{-\pi|\epsilon|/2}\,\Gamma(2\nu + 2)}{|\Gamma(\nu + 1 - i\epsilon)|} \rho_a e^{+i(\zeta+\epsilon \ln 2|\zeta|+\phi_a-\phi_\Gamma/2-\pi\sigma_\omega(\nu+1)/2)} \tag{5.1.30}$$

We can now solve for our solution that is outgoing at infinity

$$(\text{constant})\ \phi^\nu_{c,(\text{out at } \infty)} = \frac{-A}{(\sigma_\omega)^{2\nu+1}} \phi^\nu_c + \phi^{-\nu-1}_c . \tag{5.1.31}$$

The overall constant in front is irrelevant and thus we have

$$\gamma^\nu_c = -\frac{A}{(\sigma_\nu)^{2\nu+1}} = -\frac{e^{-i\pi\sigma_\omega(\nu+1/2)}}{(\sigma_\omega)^{2\nu+1}} \frac{\sin \pi(\nu + i\epsilon)\ |\Gamma(\nu + 1 \pm i\epsilon)|^2}{\sin 2\pi\nu\, \Gamma(2\nu + 1)\Gamma(2\nu + 2)} . \tag{5.1.32}$$

When $\omega > 0$, $\sigma_\omega = 1$ and our $\gamma^\nu_c = \tilde{\gamma}_\nu$ of Eq. (A.4) from [5].

The Ingoing Solution at the Horizon

The solution ϕ^ν_c is a series of confluent hypergeometric functions Eq. (5.1.9). Unfortunately, this series is not convergent as $r \to 2M$, and therefore is unsuitable for exploring the detailed behavior near the horizon. To address this problem, MST introduced a second set of solutions written as a series of standard hypergeometric equations, ${}_2F_1(a,b,c,x)$. Remarkably, in the region where both this series and the confluent hypergeometric series solution Eq. (5.1.9) converge, the solutions are the same, modulo an overall constant, K_ν,

$$\phi_c^\nu = \frac{\Gamma(2\nu+2)}{\Gamma(\nu+1+i\epsilon)} R^\nu_{c\,(\mathrm{LFE})} = \frac{\Gamma(2\nu+2)}{\Gamma(\nu+1+i\epsilon)} \frac{1}{K_\nu} R^\nu_{o\,(\mathrm{LFE})} \tag{5.1.33}$$

$$\phi_c^{-\nu-1} = \frac{\Gamma(-2\nu)}{\Gamma(-\nu+i\epsilon)} R^{-\nu-1}_{c\,(\mathrm{LFE})} = \frac{\Gamma(-2\nu)}{\Gamma(-\nu+i\epsilon)} \frac{1}{K_{-\nu-1}} R^{-\nu-1}_{o\,(\mathrm{LFE})}\,, \tag{5.1.34}$$

where $R_{c\,(\mathrm{LFE})}$ and $R_{o\,(\mathrm{LFE})}$ are the solutions given by MST. In terms of these function, the solution that is in-goin at the horizon R_{in} has the has a simple form

$$R^\nu_{\mathrm{in}} = R^\nu_{o\,(\mathrm{LFE})} + R^{-\nu-1}_{o\,(\mathrm{LFE})} \tag{5.1.35}$$

$$= \frac{\Gamma(\nu+1+i\epsilon)}{\Gamma(2\nu+2)} K_\nu \left[\phi_c^\nu + \frac{\Gamma(-\nu+i\epsilon)}{\Gamma(\nu+1+i\epsilon)} \frac{\Gamma(2\nu+2)}{\Gamma(-2\nu)} \frac{K_{-\nu-1}}{K_\nu} \phi_c^{-\nu-1}\right] \tag{5.1.36}$$

This allows us to read off the relevant coefficient for the ingoing solutions

$$\beta_c^\nu = \frac{\Gamma(2\nu+2)\Gamma(-\nu+i\epsilon)}{\Gamma(-2\nu)\Gamma(\nu+1+i\epsilon)} \frac{K_{-\nu-1}}{K_\nu} \tag{5.1.37a}$$

$$= \frac{\Gamma(2\nu+2)\Gamma(2\nu+1)}{|\Gamma(\nu+1\pm i\epsilon)|^2} \frac{\sin 2\pi\nu}{\sin\pi(\nu-i\epsilon)} \frac{K_{-\nu-1}}{K_\nu} \tag{5.1.37b}$$

$$= -(2\epsilon)^{2\nu+1} \frac{|\Gamma(\nu+1\pm i\epsilon)|^4}{\Gamma(2\nu+2)\Gamma(2\nu+1)} \left(\frac{\rho_\mu}{\rho_\lambda}\right)^2 \times \left[\frac{\cos(2\pi\nu)\ \cosh(2\pi\epsilon)-1}{2\pi\sin 2\pi\nu} - i\,\frac{\sinh 2\pi\epsilon}{2\pi}\right] \tag{5.1.37c}$$

where ρ_μ and ρ_λ come from the summations in the definition of K_ν. After some simplification, they can be written as

$$\lambda^\nu = \sum_{n=0}^{\infty} (-1)^n \frac{(2\nu+1)_n}{n!} \overline{a_n^\nu} \quad \text{and} \quad \rho_\lambda = |\lambda^\nu| \tag{5.1.38}$$

$$\mu^\nu = \sum_{n=0}^{\infty} (-1)^n \frac{(-2\nu-1)_n}{n!} a_n^{-\nu-1} \quad \text{and} \quad \rho_\mu = |\mu^\nu|\ . \tag{5.1.39}$$

Using the fact that $a_n^n = a_{-n}^{-\nu-1}$, we have $\mu^{-\nu-1} = \overline{\lambda^\nu}$. These definitions for β_c^ν agree with $\widetilde{\beta}_\nu$ from Eq. (A.3) in [5].

In order to solve for our retarded field, we will expand all functions in terms of the two parameters $\epsilon = 2M\omega$ and $z = \omega r$. This is similar to a pN expansion because $z \sim v$ and $\epsilon/z = 2M/r = 2v^2$ (for geodesic motion), and thus we will frequently refer to expressions in terms of their pN order. When we do this, it should be understood that these terms are only first order in the mass ratio.

To understand this approach, it is necessary to study the individual elements in ϕ_c^ν more closely. If we rewrite Eq. (5.1.18) in terms of only z and ϵ, we find,

$$\begin{aligned}\phi_c^\nu = (2z)^\nu \Phi^\nu &= (2z)^\nu \left(1 - \frac{\epsilon}{z}\right)^\nu e^{-iz(1-\epsilon/z)} \sum_{n=-\infty}^{\infty} a_n^\nu (2iz)^n \left(1 - \frac{\epsilon}{z}\right)^n \\ &\quad \times \frac{(\nu + 1 + i\epsilon)_n}{(2\nu+2)_{2n}} {}_1F_1(n + \nu + 1 + i\epsilon, 2n + 2\nu + 2, 2iz(1 - \epsilon/z)) \, .\end{aligned} \tag{5.2.1}$$

In solving for the a_n^ν we find that $\nu = \ell + O(\epsilon^2)$, which allows us to simply expand $(2z)^\nu \left(1 - \frac{\epsilon}{z}\right)^\nu$ about $\epsilon = z = 0$. It quickly becomes apparent that the function Φ^ν can be written as a double power series in ϵ/z and z^2.

From Eqs. (5.1.15), it is clear that for any given ℓ value for $n \geq 0$, $a_n^\nu \sim \epsilon a_{n-1}^\nu$. If we combine this with the $2iz^n$ term, then we know that for $n \geq 0$, each term is led by a term proportional to $(\epsilon z)^n = z^{2n}(\epsilon/z)^n$. Therefore, as a pN expansion, each n-mode of the sum is led by a term proportional to v^{4n}, meaning that for any practical calculation, the upper limit on the sum will be a small finite number, as all higher n-modes will be of too high order to contribute.

Similarly for $n < 0$, $(2iz)^n a_n^\nu \sim (\epsilon/z)^n \sim v^2 n$. For these cases, it is important to consider the special cases when the L_n are not proportional to ϵ. These occur for low ℓ modes and need to be handled on a case by case basis. Even in these special cases, it is possible to write the functions as a series in ϵ/z and z^2.

Therefore, our sum is no longer over an infinite number of terms but instead a relatively small number of terms. The ratio of Pochhammer symbols in Eq. (5.2.1) is a ratio of two low-order polynomials in ϵ. For larger values of n, the polynomials grow larger, but we require fewer terms from their expansions due to the factors of v^2 provided by the $a_n^\nu(2iz)^n$.

Last, we consider the hypergeometric function, ${}_1F_1(n+\nu+1+i\epsilon, 2n+2\nu, 2iz(1-\epsilon/z)$. Using Eq. (13.1.2) from [58],

$$ {}_1F_1(a,b,x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k}\frac{x^k}{k!}, \tag{5.2.2} $$

we can rewrite the confluent hypergeometric functions in terms of a series in ϵ/z and z^2 as well.

In doing this, it is worth noting that Φ^ν is a regular polynomial in the expansion parameters, and we only find terms proportional to $\ln[v]$ when we include $(2z)^\nu$. This feature will play a crucial role in the specialization of Hikida et al. [4] we introduce in the next section.

In practice, it is useful to introduce a "smallness" parameter λ, and make the substitution $z \to \lambda z$, $\omega \to \lambda\omega$ and $\epsilon \to \lambda^3\epsilon$. This allows us to generate a series expansion in a single parameter, λ.

One natural objection to this method is that when we consider accelerated orbits we cannot make use of Kepler's law, which tells us that $\epsilon/z \sim z^2$. To demonstrate the potential difficulties in this scenario, we will write the particle's accelerated velocity as $v = v_{geo} + \delta v$ so that we can write $v^2 = (1+\alpha)^2 v_{geo}^2$, where $\alpha = \delta v/v$. Now, if we consider the case where the smaller black hole is moving in a circular orbit at $r = 10^6 M$, then the particle's "geodesic" speed would be $v_{geo} = 0.001$. Now, let us accelerate the particle so that it is moving at $v = 100 v_{geo} = 0.1$. Any terms of order $\epsilon/z = 2M/r$ would be proportional to v_{geo}^2. On the other hand, terms proportional to v^2 would be proportional to $10^4 v_{geo}^2$, meaning that terms which are comparable in magnitude for geodesic orbits may be of completely different magnitudes for accelerated orbits.

Despite this complication, we will still proceed by treating v_{geo} as our small parameter (or equivalently $\sqrt{M/r}$). We do this for several reasons. First of all, the main point in considering accelerated orbits is to disentangle the effects of the particle's speed from those of the spacetime curvature, so that we can gain a greater understanding of what effects come into play for particles traveling along geodesics. Therefore, we want to treat terms that are of equal magnitude for geodesic orbits on the same footing as our accelerated orbits.

By expanding about $v_{geo} = 0$, we do introduce some potential difficulties in interpreting numerical results. Consider the previous example of the particle accelerated to move 100 times faster than it would were it following a geodesic at $r = 10^6 M$. If we state that we are collecting a pN 6 term, we do not get a term of the same order as $(M/r)^6$, as we would in the case of geodesic orbits. Instead, we get a term of order $v^{12} = 10^{24} v_{geo}^{12}$, so that, when we analyze our answer, we can only trust terms out to roughly twelve decimal places, which is to say that the expansion is through v^{12} and $(M/r)^2$. Similarly, if we slowed the particle down so that it moved at $v_{geo}/100$, then $v^{12} = 10^{-24} v_{geo}^{12}$ so we obtain the same accuracy as we would expect for a geodesic orbit. So, this time, the expression would be of order $(M/r)^6$ and v^7.

In short, we will consider an expansion in z and ϵ, where we will think of $\epsilon \approx O(z^3)$. We will consider terms to be of the same pN order if the magnitudes of the terms are of the same order when $v \to v_{geo}$. For analyzing the accuracy of any numerical results for accelerated particles, we

5.3 Green's Functions

Now that we have discussed the retarded solutions to the field equations, our goal is to use these solutions to obtain the regularized self-force on the particle. To accomplish this task, we will first need to consider the Green's functions themselves (in section 5.3.1), wherein we discuss a useful splitting of the fields introduced by Hikida et al. [4], where they split the Green's function into two pieces, the so-called $\tilde{R}$ piece and the $\tilde{S}$ piece. We will treat these two pieces separately, as the two functions have very different properties, and first compute the $\tilde{R}$ contribution to the force and then we will consider the $\tilde{S}$ contribution to the force.

5.3.1 Green's Function

Now that we have discussed the source-free solutions, we generate our Green's function

$$g_{\omega\ell m}(r_<, r_>) = -\frac{\phi^\nu_{in}(r_<)\phi^\nu_{up}(r_>)}{W_{\omega\ell m}(\phi^\nu_{in}, \phi^\nu_{up})}, \tag{5.3.1}$$

where

$$W_{\omega\ell m}(\phi^\nu_{in}, \phi^\nu_{up}) = r^2\left(1 - \frac{2M}{r}\right)\left[(\partial_{r_>} - \partial_{r_<})\phi^\nu_{in}(r_<)\phi^\nu_{up}(r_>)\right]_{r_<=r_>}. \tag{5.3.2}$$

The solutions ϕ^ν_{in} and ϕ^ν_{up} are the ingoing solution at the event horizon of the super massive black hole and the out-going solution at infinity respectively. These can be related to the solutions ϕ^ν_c and $\phi^{-\nu-1}_c$ by Eq. (2.8) in [5]

$$\begin{aligned} \phi^\nu_{in} &= \phi^\nu_c + \beta^\nu_c \phi^{-\nu-1}_c \\ \phi^\nu_{up} &= \gamma^\nu_c \phi^\nu_c + \phi^{-\nu-1}_c \end{aligned} \tag{5.3.3}$$

We now write the Green's function as the sum of two pieces,

$$g_{\ell m\omega}(r_<, r_>) = g^{\tilde{R}}_{\ell m\omega}(r_<, r_>) + g^{\tilde{S}}_{\ell m\omega}(r_<, r_>), \tag{5.3.4}$$

where

$$\begin{aligned} g^{\tilde{R}}_{\ell m\omega}(r_<, r_>) &= \frac{-1}{(1-\gamma^\nu_c\beta^\nu_c)W_{\ell m\omega}}[\gamma^\nu_c\phi^\nu_c(r_<)\phi^\nu_c(r_>) + \beta^\nu_c\phi^{-\nu-1}_c(r_<)\phi^{-\nu-1}_c(r_>) \\ &+ \beta^\nu_c\gamma^\nu_c(\phi^\nu_c(r_<)\phi^{-\nu-1}_c(r_>) + \phi^\nu_c(r_>)\phi^{-\nu-1}_c(r_<))], \end{aligned} \tag{5.3.5}$$

and

$$g^{\tilde{S}}_{\ell m\omega}(r_<, r_>) = \frac{-1}{W_{\ell m\omega}}\phi^\nu_c(r_<)\phi^{-\nu-1}_c(r_>). \tag{5.3.6}$$

For now let us focus only on the $\tilde{R}$ piece of the Green's function, as this is the piece required for the damping force. We will save a discussion of the $\tilde{S}$ piece for next Chapter.

Using our knowledge of the behavior of these solutions from section 5.2, we will demonstrate that in order to compute the contributions for a given (finite) pN order we need compute only a finite number of ℓ terms, as all of the other terms will be of too high a pN order. We shall make reference to Eqs. (5.1.18), (5.1.32), and (5.1.37c) in conjunction with Eq. (5.3.5).

Once again, let us consider the high-ℓ terms so that we need not concern ourselves with the behavior of the a_n^ν coefficients for the special values of n. Recall that $\phi_c^\nu = (2z)^\nu \Phi^\nu$ and that Φ^ν is regular at $\epsilon = 0$ and $z = 0$. then we can rewrite Eq. (5.3.5) as

$$g_{\ell m\omega}^{\tilde{R}} \sim \frac{-1}{(1-\gamma_c^\nu \beta_c^\nu) W_{\ell m\omega}} \left[\gamma_c^\nu (2z)^{2\nu} + \beta_c^\nu (2z)^{-2\nu-2} + \frac{\beta_c^\nu \gamma_c^\nu}{z} \right] (1 + O(\epsilon/z, z^2)). \tag{5.3.7}$$

Now, given that $\gamma_c^\nu \sim \epsilon^{-1}$, $\beta_c^\nu \sim \epsilon^{2\nu+1}$, $W_{\ell m\omega} \sim (1 + O(\epsilon^2))/\omega$, and $\nu = \ell + O(\epsilon^2)$, we can write

$$g_{\ell m\omega}^{\tilde{R}} \sim \frac{-(1 + O(\epsilon/z, z^2))}{r(1-\epsilon^{2\ell})} \left[\frac{(2z)^{2\ell+1}}{\epsilon} + \left(\frac{\epsilon}{2z}\right)^{2\ell+1} + \epsilon^{2\ell} \right] (1 + O(\epsilon/z, z^2)). \tag{5.3.8}$$

For large ℓ, the first term in the brackets will dominate. By using $z \sim v$, it is clear that $g_{\ell m\omega}^{\tilde{R}} \sim (v^2)^{\ell-1}$. Therefore, to achieve N pN orders of accuracy, it is necessary to compute $N+1$ ℓ-modes of $g_{\ell m\omega}^{\tilde{R}}$.

For practical calculations, this indicates that the contributions for the $\tilde{R}$ piece of the field fall off faster than any power of ℓ. [6]

5.4 Solving for the retarded field.

5.4.1 General ℓ Solutions

We will now solve for Φ^ν and $\Phi^{-\nu-1}$ for general ℓ. These expressions will not necessarily hold for small ℓ. If we want to write an expression that is valid to pN order 'N' then we need to compute the modes $\ell = 0$ through $\ell = N+1$ explicitly. The rest of the modes are correctly described by the general ℓ expression. We will borrow the shorthand from Hikida et al. [5] and say that this expression is valid for $\ell > pN+1$.

To understand why this expression is limited to $\ell > pN+1$, recall Eq. (5.1.16), where we see that each of the left movers (and thus each of the a_n^ν for $n < 0$) break from the typical behavior for certain values of n. So, for example, with $\ell = 10$, we can safely

[6] While it is probably true that one can show that the fields from $g_{\ell m\omega}^{\tilde{R}}$ are in fact C^∞ in general, we will make the slightly weaker claim that for finite pN order the approximations are C^∞.

generate a^{ν}_{-10} before we encounter the irregular behavior, where as for $\ell = 0$, the irregular behavior appears already at a^{ν}_{-1}.

We solve for the a^{ν}_{n} and use the three term recurrence relation for $n = 0$ to solve for ν. If we define

$$\nu = \ell + \sum_{k=1}^{\infty} \nu_{2k}\epsilon^{2k}, \tag{5.4.1}$$

then we can write

$$\nu_2 = \frac{-15\ell^2 - 15\ell + 11}{2(2\ell+1)[(2\ell+3)(2\ell-1)]}, \tag{5.4.2}$$

$$\begin{aligned} \nu_4 &= \frac{-1}{8\ell(\ell+1)(2\ell+1)^3[(2\ell+3)(2\ell-1)]^3[(2\ell+5)(2\ell-3)]} \\ &\times (3240 + 8733\ell - 73892\ell^2 - 9955\ell^3 + 278260\ell^4 + 64365\ell^5 \\ &- 382305\ell^6 - 235200\ell^7 + 79800\ell^8 + 92400\ell^9 + 18480\ell^{10}). \end{aligned} \tag{5.4.3}$$

For the work we report, on these two corrections are sufficient, but we give the next correction here as well,

$$\begin{aligned} \nu_6 &= \frac{\left([(2\ell+5)(2\ell-3)]^2[(2\ell+7)(2\ell-5)]\right)^{-1}}{16(\ell^2(\ell+1)^2(\ell-1)(\ell+2)(2\ell+1)^5[(2\ell+3)(2\ell-1)]^5} \\ &\times(112266000 + 148424400\ell - 2435958990\ell^2 - 6168553647\ell^3 \\ &+354780315264\ell^4 + 363889459295\ell^5 - 196940982399\ell^6 - 140591485296\ell^7 \\ &+553770389547\ell^8 + 435348291492\ell^9 - 815344024118\ell^{10} \\ &-859441621500\ell^{11} + 504925684186\ell^{12} + 867198262392\ell^{13} \\ &+23676278472\ell^{14} - 393024360960\ell^{15} - 143963649984\ell^{16} \\ &+59163616512\ell^{17} + 47111896832\ell^{18} + 4750986240\ell^{19} \\ &-3550170624\ell^{20} - 1150076928\ell^{21} - 104552448\ell^{22}). \end{aligned} \tag{5.4.4}$$

The Wronskian, $W_{\ell m\omega}$ is

$$W_{\ell m\omega} = -\frac{2\ell+1}{2\omega} + \frac{(131 - 2\ell(1+\ell)(175 + 8\ell(1+\ell)(-17 + 2\ell(1+\ell))))}{8\omega(2\ell+1)[(2\ell+3)(2\ell-1)]^2}\epsilon^2$$
$$+\frac{\epsilon^4}{32\omega\ell^2(\ell+1)^2(2\ell+1)^3[(2\ell+3)(2\ell-1)]^4[(2\ell+5)(2\ell-3)]^2}$$
$$\times\{291600 - \ell(1+\ell)(-1205280 + \ell(1+\ell)(4159782$$
$$+\ell(1+\ell)[16861932 + \ell(1+\ell)(-78712065 + 2\ell(1+\ell)(61438379$$
$$+4\ell(1+\ell)[-11436715 + 8\ell(1+\ell)(494407 + 8\ell(1+\ell)(-9162$$
$$+\ell(1+\ell)(343 + 20\ell(1+\ell))))])])))\}$$
$$+O(\epsilon^6). \tag{5.4.5}$$

We will write the solutions Φ^ν as

$$\Phi^{\nu/-\nu-1} = \sum_{n,m} C^{\nu/-\nu-1}_{n,2m}\omega^{2m}\left(\frac{M}{r}\right)^{\frac{2n-2m}{2}} \tag{5.4.6}$$

where the pN order is described by n and the power of ω is given by m. With these definitions, and recalling that $z = \omega r$ and $\epsilon = 2M\omega$. Thus, to 6 pN order,

$$\Phi^\nu = 1 - \ell\frac{M}{r} - \frac{\omega^2r^2}{2(2\ell+3)} + \frac{\ell^2-5\ell-10}{2(2\ell+3)(\ell+1)}\left(\frac{M}{r}\right)(\omega r)^2 + \frac{(\omega r)^4}{8(2\ell+3)(2\ell+5)}$$
$$+\frac{\ell(\ell-1)^2}{2\ell-1}\left(\frac{M}{r}\right)^2 - \frac{\ell(\ell-1)(\ell-2)^2}{3(2\ell-1)^2}\left(\frac{M}{r}\right)^3$$
$$-\frac{\ell^3-18\ell^2+17\ell-4}{2(2\ell-1)^2}\left(\frac{M}{r}\right)^2(\omega r)^2$$
$$-\frac{3\ell^3-27\ell^2-142\ell-136}{24(\ell+1)(\ell+2)(2\ell+3)(2\ell+5)}\left(\frac{M}{r}\right)(\omega r)^4$$
$$-\frac{(\omega r)^6}{48(2\ell+3)(2\ell+5)(2\ell+7)}$$
$$+\sum_{n=4}^{6}\sum_{m=0}^{n} C^\nu_{n,2m}(\omega r)^{2m}\left(\frac{M}{r}\right)^{\frac{2n-2m}{2}} + 7pN, \tag{5.4.7}$$

where the 4th pN terms are given by

$$
\begin{aligned}
C^{\nu}_{4,0} &= \frac{\ell(\ell-1)(\ell-2)^2(\ell-3)^2}{6(2\ell-3)(2\ell-1)}, \\
C^{\nu}_{4,2} &= \frac{2\ell^6 - 61\ell^5 + 53\ell^4 + 386\ell^3 - 286\ell^2 - 4\ell + 24}{6\ell(2\ell+1)(2\ell-1)^2}, \\
C^{\nu}_{4,4} &= \frac{1}{24(\ell+1)(\ell+2)(2\ell-1)^2(2\ell+1)(2\ell+3)^3(2\ell+5)} \\
&\quad \times \Big(48\ell^9 - 1152\ell^8 - 7040\ell^7 - 8212\ell^6 + 10953\ell^5 \\
&\quad +15745\ell^4 - 10867\ell^3 - 7749\ell^2 + 6930\ell^1 - 768 \Big), \\
C^{\nu}_{4,6} &= \frac{5\ell^4 - 60\ell^3 - 625\ell^2 - 1548\ell - 1108}{240(\ell+1)(\ell+2)(\ell+3)(2\ell+3)(2\ell+5)(2\ell+7)}, \\
C^{\nu}_{4,8} &= \frac{1}{384(2\ell+3)(2\ell+5)(2\ell+7)(2\ell+9)},
\end{aligned}
\tag{5.4.8}
$$

and the 5th pN terms are

$$
\begin{aligned}
C^{\nu}_{5,0} &= -\frac{\ell(\ell-1)(\ell-2)(\ell-3)^2(\ell-4)^2}{30(2\ell-3)(2\ell-1)} \\
C^{\nu}_{5,2} &= -\frac{1}{12\ell(2\ell-3)(2\ell-1)^2(2\ell+1)(2\ell+3)} \\
&\quad \times \Big(4\ell^9 - 188\ell^8 + 483\ell^7 + 3127\ell^6 - 6795\ell^5 - 4211\ell^4 \\
&\quad +13208\ell^3 - 4404\ell^2 - 936\ell + 432 \Big) \\
C^{\nu}_{5,4} &= -\frac{1}{24\ell(\ell+1)^2(2\ell-1)^2(2\ell+1)(2\ell+3)^3(2\ell+5)} \\
&\quad (16\ell^{11} - 768\ell^{10} - 672\ell^9 + 31236\ell^8 + 169443\ell^7 + 405867\ell^6 + 453521\ell^5 + \\
&\quad 67017\ell^4 - 278316\ell^3 - 115776\ell^2 + 59568\ell + 6480), \\
C^{\nu}_{5,6} &= -\frac{1}{240(\ell+1)(\ell+2)(\ell+3)(2\ell-1)^2(2\ell+1)(2\ell+3)^3(2\ell+5)^2(2\ell+7)} \\
&\quad \times (160\ell^{11} - 5200\ell^{10} - 53840\ell^9 - 74872\ell^8 + 715258\ell^7 + 3065539\ell^6 \\
&\quad +4173300\ell^5 + 569492\ell^4 - 2743668\ell^3 - 883399\ell^2 + 690870\ell + 37080), \\
C^{\nu}_{5,8} &= -\frac{35\ell^5 - 490\ell^4 - 8855\ell^3 - 40754\ell^2 - 73032\ell - 43968}{13440(\ell+1)(\ell+2)(\ell+3)(\ell+4)(2\ell+3)(2\ell+5)(2\ell+7)(2\ell+9)}, \\
C^{\nu}_{5,10} &= -\frac{1}{3840(2\ell+3)(2\ell+5)(2\ell+7)(2\ell+9)(2\ell+11)},
\end{aligned}
\tag{5.4.9}
$$

and the 6th pN terms are

$$
\begin{aligned}
C^{\nu}_{6,0} &= \frac{\ell(\ell-1)(\ell-2)(\ell-3)^2(\ell-4)^2(\ell-5)^2}{90(2\ell-5)(2\ell-3)(2\ell-1)}, \\
C^{\nu}_{6,2} &= \frac{1}{60\ell(\ell-1)(2\ell-3)(2\ell-1)^2(2\ell+1)(2\ell+3)} \\
&\quad \times(4\ell^{11} - 272\ell^{10} + 1775\ell^9 + 3720\ell^8 - 40838\ell^7 + 70264\ell^6 \\
&\quad +28955\ell^5 - 167960\ell^4 + 126504\ell^3 - 14232\ell^2 - 12000\ell + 2880), \\
C^{\nu}_{6,4} &= \frac{1}{144\ell^2(2\ell-3)^2(2\ell-1)^4(2\ell+1)(2\ell+3)} \\
&\quad \times(48\ell^{12} - 4224\ell^{11} + 64064\ell^{10} - 95940\ell^9 - 379631\ell^8 \\
&\quad +791789\ell^7 + 698871\ell^6 - 2756237\ell^5 + 2223936\ell^4 \\
&\quad -321444\ell^3 - 418608\ell^2 + 215136\ell - 31104), \\
C^{\nu}_{6,6} &= \frac{1}{720\ell(\ell+1)^2(\ell+2)^2(\ell+3)(2\ell-1)^2(2\ell+1)(2\ell+3)^3(2\ell+5)^2(2\ell+7)} \\
&\quad \times(+160\ell^{15} - 10000\ell^{14} - 51120\ell^{13} + 1059592\ell^{12} + 14222570\ell^{11} \\
&\quad +85301127\ell^{10} + 323138174\ell^9 + 837487174\ell^8 + 1451995896\ell^7 \\
&\quad +1484991533\ell^6 + 500977876\ell^5 - 576841506\ell^4 - 577239396\ell^3 \\
&\quad -27215040\ell^2 + 99958320\ell + 6804000), \\
C^{\nu}_{6,8} &= \frac{[(2\ell+1)(2\ell+3)^3(2\ell+5)^2(2\ell+7)^2(2\ell+9)]^{-1}}{40320(\ell+1)(\ell+2)(\ell+3)(\ell+4)(2\ell-1)^2} \\
&\quad \times(+6720\ell^{13} - 262080\ell^{12} + -4564560\ell^{11} - 12655776\ell^{10} \\
&\quad +155479612\ell^9 + 1411722500\ell^8 + 5057129549\ell^7 + 9094631456\ell^6 \\
&\quad +6927573308\ell^5 - 1535261710\ell^4 - 5056201229\ell^3 - 1011515670\ell^2 \\
&\quad +1028051640\ell + 110557440), \\
C^{\nu}_{6,10} &= \frac{[(2\ell+3)(2\ell+5)(2\ell+7)(2\ell+9)(2\ell+11)]^{-1}}{80640(\ell+1)(\ell+2)(\ell+3)(\ell+4)(\ell+5)} \\
&\quad \times\Big(21\ell^6 - 315\ell^5 - 9205\ell^4 - 67921\ell^3 \\
&\quad -219992\ell^2 - 323836\ell - 172976\Big), \\
C^{\nu}_{6,12} &= \frac{1}{40680(2\ell+3)(2\ell+5)(2\ell+7)(2\ell+9)(2\ell+11)(2\ell+13)}.
\end{aligned}
\tag{5.4.10}
$$

Comparing with Hikida et al. [4], we agree with all of the terms they reported (the first line of Eq. (5.4.7)). All of the other terms are new.

To get $\Phi^{-\nu-1}$ we can simply make the substitution $\ell \leftrightarrow -\ell-1$. All of these values assume that the denominators of α_n^ν, γ_n^ν, and the leading order term of β_n^ν do not vanish. Therefore, for lower values of $\ell \leq 7$, one needs to solve for the Φ^ν and $\Phi^{-\nu-1}$, directly for each value of ℓ. [7]

5.5 The Damping force

A charged particle moving through curved spacetime interacts with its own field. The resulting force has both conservative and non-conservative pieces. We only consider the non-conservative in this chapter, and devote the next Chapter to the computation of the conservative self-force.

As the non-conservative part of the self-force is purely a damping force, it acts directly against the motion of the particle. For circular orbits, this is most convenient because this damping force is just the component of the force in the ϕ direction. Therefore, we can write

$$F_\phi = i\frac{q^2}{u^t}\sum_{\ell,m} m g_{\ell,m,m\Omega}(r_0,r_0)|Y_{\ell m}(\pi/2,0)|^2. \tag{5.5.1}$$

Now, we will break $g_{\ell,m,m\Omega}$ into the $\tilde{R}$ and $\tilde{S}$ pieces. However, we can immediately see that $F_\phi^{\tilde{S}} = 0$ because $g^{\tilde{S}}_{\ell,m,m\Omega}(r_<,r_>)$ is a real function that is even in ω, which in turn tells us that when we perform the appropriate sum in Eq. (5.5.2), each term will be of the form $m^{2n+1}|Y_{\ell,m}(\pi/2.0)|^2$, which vanishes when summed over m. This allows us to write

$$F_\phi = F_\phi^{\tilde{R}} = i\frac{q^2}{u^t}\sum_{\ell,m} m g^{\tilde{R}}_{\ell,m,m\Omega}(r_0,r_0)|Y_{\ell m}(\pi/2,0)|^2. \tag{5.5.2}$$

Using the knowledge that $u^t = (1-2M/r-(\Omega r)^2)^{-1/2}$, we can write down the damping

[7] By examining Eq. (5.4.7) one can already see evidence of this behavior. In the term proportional to $M\omega^4$, when $\ell \to -\ell-1$, the denominator becomes to $\ell(\ell-1)(2\ell-1)(2\ell-3)$, which obviously vanishes for $\ell = 0$ and $\ell = 1$. A comparison with the value from the explicit terms shows that this does not agree with that found for $\ell = 2$ but it does for $\ell \geq 3$.

force as

$$
\begin{aligned}
F_\phi &= -\frac{q^2\Omega^2}{4\pi r_0^2}\Bigg[\left(\frac{r_0^2\Omega^2}{3}\right)+\left(\frac{5r_0^4\Omega^4}{6}-Mr\Omega^2\right)+\left(\frac{2\pi}{3}Mr_0^2|\Omega^3|\right)\\
&+\left(\frac{35r_0^6\Omega^6}{24}-\frac{11r_0^3\Omega^4M}{2}+\frac{5(M\Omega)^2}{6}\right)+\left(\frac{19\pi r_0^4|\Omega|^5M}{5}-2\pi r_0|\Omega|^3M^2\right)\\
&+\Bigg(-M^2r_0^2\Omega^4\left(\frac{76}{45}(\ln[2\Omega r_0]+\gamma)-\frac{4\pi^2}{9}-\frac{46537}{2700}\right)-\frac{M^3\Omega^2}{6}+\frac{35r_0^8\Omega^8}{16}\\
&-\frac{19201Mr_0^5\Omega^6}{1080}+\frac{4M^4}{3r_0^4}\Bigg)+\Bigg(\frac{4639\pi r_0^6|\Omega^7|M}{420}-\frac{65\pi M^2r_0^3|\Omega^5|}{3}\\
&+\frac{5\pi|\Omega^3|M^3}{3}\Bigg)+\Bigg(\frac{385r_0^{10}\Omega^{10}}{128}-\frac{3215311r_0^7\Omega^8M}{75600}-r_0^4\Omega^6M^2\Bigg\{\frac{-335959619}{2646000}\\
&+\frac{18362}{1575}(\gamma+\ln[2\Omega r_0]))-\frac{242\pi^2}{45}+\frac{20224}{1575}\ln[2]\Bigg\}+M^3r_0\Omega^4\Bigg(-\frac{20417}{900}\\
&-\frac{4}{3}\pi^2+\frac{76}{15}(\gamma+\ln[2\Omega r_0])\Bigg)+\frac{5\Omega^2M^4}{8}+\frac{4M^5}{r_0^5}\Bigg)+\Bigg(\frac{546307\pi r_0^8|\Omega^9|M}{22680}\\
&-\frac{11675\pi r_0^5|\Omega^7|M^2}{108}-\frac{\pi|\Omega^3|M^4}{3r_0}-\pi M^3r_0^2|\Omega|^5\Bigg(-\frac{71977}{1350}\\
&+\frac{152}{15}(\gamma+\ln[2\Omega r_0])\Bigg)\Bigg)+7pN\Bigg] \qquad (5.5.3)
\end{aligned}
$$

For geodesic orbits, this force is much simpler. Using Kepler's law, we write $M/r = (\Omega r)^2 = v^2$, where v is the orbital velocity.

$$
\begin{aligned}
F_\phi^{geo} &= -\frac{q^2v^4}{4\pi r_0^2}\Bigg[\frac{1}{3}-\frac{v^2}{6}+\frac{2\pi|v|^3}{3}-\frac{77v^4}{24}+\frac{9\pi|v^5|}{5}\\
&+\frac{10121+1600\pi^2-6080(\gamma+\ln[2v])}{3600}v^6-\frac{3761\pi|v^7|}{420}\\
&+\frac{489584469+28537600\pi^2-90603520\ln[2]-46511360(\gamma+\ln[2v])}{7056000}v^8\\
&-\pi\frac{3518947+383040(\gamma+\ln[2v])}{113400}|v^9|\Bigg]+7pN \qquad (5.5.4)
\end{aligned}
$$

To arrive at Eqs. (5.5.3) and (5.5.4) it is only necessary to use $\ell = 0$ through $\ell = 7$. All higher ℓ terms are of too high a pN order to contribute.

Chapter 6

The Conservative Self-Force

In computing the conservative self-force, we must renormalize our fields by subtracting the contributions from the singular field. In doing this, we will need to focus on many of the subtleties that we were able to push aside before: To this point we have only developed the background for regularization, but have yet to perform an actual regularization.

In this Chapter, we will demonstrate some remarkable results, results which could ease the computational burden of self-force calculations considerably. This benefit was noted by Hikida et al. [4], and indeed was the primary focus of their papers. Unfortunately, it seems that these methods have either been ignored or are unknown to many in the self-force field.

6.1 The $\tilde{S}$ and $\tilde{R}$ fields and Detweiler and Whiting's S and R fields

The labeling chosen by Hikida et al. [4], calling the solution to the source-free field equations the $\tilde{R}$ field and to call the solution to the sourced equations the $\tilde{S}$ field, is an intentional comparison to the R and S fields of Detweiler and Whiting [36]. As we are going to regularize these fields now, it is important to understand the differences between these four fields.

As discussed in chapters 2 and 3, the DW singular field, ϕ^S, is chosen so that it can be subtracted straight from the retarded field, ϕ^{ret} so that the resulting field, ϕ^R is a smooth, C^∞ function *in the normal neighborhood.* To define the S and R fields beyond the normal neighborhood, we would need to choose a manner of extending these fields to the rest of the spacetime, and, importantly, choose an extension which *does not change the self-force renormalization*[1].

The $\tilde{S}$ and $\tilde{R}$ fields have some similar properties to the S and R fields respectively, but they are different in a few important ways. First of all, the $\tilde{S}$ and $\tilde{R}$ fields are both defined globally, so there is no need to consider extending them to a certain region. The function $\phi^{\tilde{R}}$ (restricted to a finite pN order) is C^∞ in its entire domain, just as the ϕ^R field is in its (considerably smaller) domain, as both functions are solutions to the source-free field equations.

Similarly, $\phi^{\tilde{S}}$ and ϕ^S are solutions to the same sourced field equations with $\tilde{S}$ defined globally, not locally. However, $\phi^{\tilde{S}}$ is *not* a globally defined singular field. When one subtracts these fields from each other, the resulting $\phi^{\tilde{S}-S} = \phi^{\tilde{S}} - \phi^S$ field has a nonvanishing contribution to the self-force. The renormalized field is given by

$$\phi^R = \phi^{\tilde{R}} + (\phi^{\tilde{S}} - \phi^S). \tag{6.1.1}$$

6.2 The $\tilde{R}$ Contribution to the Force

Consider the equation for the radial force,

$$F_r^{ret} = \frac{q^2}{u^t} \sum_{\ell,m} \partial_r g_{\ell,m,m\Omega}(r, r_0)\Big|_{r=r_0} |Y_{\ell,m}(\pi/2, 0)|^2. \tag{6.2.1}$$

Unlike the ϕ component of the force, the radial component will have contributions from both the $\tilde{R}$ and $\tilde{S}$ fields. We can understand this simply by recalling that the real parts of $g^{\tilde{R}}_{\ell,m,\omega}$ and $g^{\tilde{S}}_{\ell,m,\omega}$ are both even functions of ω so when $\omega \to m\Omega$ we will have even powers of m in the summand for both fields, which means that both will have a non-vanishing contribution.

As we discussed in the previous Chapter, to achieve accuracy to N pN order it is necessary to compute the $\ell = 0$ to $\ell = N + 1$. For all $\ell > N + 1$, the expressions are too high in pN order. In this way, we can see that the $\tilde{R}$ field falls off faster than any power

[1] which is to say, we choose a smooth extension.

of ℓ (for a given pN order), and therefore it does not need renormalization. We can gain some insight into this mathematical split by examining the first handful of pN orders

$$\begin{aligned}
\frac{F_r^{\tilde{R}}}{\frac{q^2}{4\pi r^2}} &= \left[\frac{5r^3\Omega^2}{19M}+\frac{2}{7}\right]+\left[\frac{1417r^5\Omega^4}{3002M}-\frac{89(r\Omega)^2}{133}-\frac{4M}{7r}\right] \\
&+\left[\frac{9804331r^7\Omega^6}{14205464M}-\frac{136153(r\Omega)^4}{42028}+\frac{5rM\Omega^2}{38}-\frac{9M^2}{7r^2}\right] \\
&+\left[\frac{1015083323057r^9\Omega^8}{1108452355920M}-\frac{1243509067(r\Omega)^6}{127849176}+\frac{3949\Omega^2M^2}{5054}-\frac{482M^3}{133r^3}\right. \\
&\left.-(Mr^3\Omega^4)\frac{(43348880(\gamma+\ln[2r\Omega])-281678493}{32511660}\right]+pN4. \qquad (6.2.2)
\end{aligned}$$

Just glancing at this equation is enough to recognize that this cannot be the physical force. First of all, there are terms of order M^{-1}, so the flat spacetime limit is clearly incorrect. Furthermore, the static particle limit also fails, since it has been established ([59]) that there is no self-force on a static scalar charge in Schwarzschild spacetime. This tells us that each of these terms must appear with opposite sign in the force $F_r^{\tilde{S}-S}$. Looking ahead to the results, it turns out that none of the terms below 3pN survive. In order to extract real physical insight, we need to consider the full, renormalized self-force

6.3 The Large ℓ Behavior of the $\tilde{S}$ and S fields

The Green's function $g^{\tilde{S}}_{\ell,m,\omega}(r_<, r_>)$ is a solution to the sourced field equations and is therefore singular at the position of the particle. As a result, the harmonic decomposition of $F_r^{\tilde{S}}$ will not fall off faster than any power of ℓ at the particle, but it will in fact diverge, requiring the computation of a large number of ℓ modes. In chapters 1-4, the focus has been entirely on the singular field, but now that we are actually renormalizing the force, there are a few subtleties that we need to discuss.

6.3.1 The High-ℓ Expansion of F_α^R

As we stated in chapters 2 and 3, the ℓ modes of the renormalized field can, in principle, be written such that they fall off faster than any power of ℓ. In practice, however, $F^R_{\alpha,\ell}$ does not fall off this quickly due to the presence of the $D_\alpha^{(2n)}$ terms. Recall Eq. (3.7.4), reproduced below:

$$F^S_{\alpha,\ell} = A_\alpha L + B_\alpha + \sum_{j=1}^{\infty} \frac{4^j D^{2j}_\alpha}{\prod_{k=1}^{j}[(2\ell+1+2k)(2\ell+1-2k)]}, \tag{6.3.1}$$

As we subtract away successive D^{2j} terms, we make the expression for the modes of the renormalized self-force fall off faster with each term subtracted, without changing the value of the force. If we knew all of these parameters, then the difference $F^{ret}_{\alpha,\ell} - F^S_{\alpha,\ell}$ would indeed fall off faster than any power in ℓ. In practice, however, we can only approximate $F^R_{\alpha,\ell}$, because we actually use an approximation for the singular field, we can therefore only approximate F^R_α.

Because of this difficulty, Heffernan et al. [31] computed several of these parameters, noting how at $\ell_{max} = 50$ the inclusion of $D^{(2)}_\alpha$ gave a relative error of only 10^{-9}, and including yet higher order parameters sees further improvement (by including the first three parameters, the relative error was 10^{-17}).

By following the splitting of the field introduced by Hikida et al. [4], we found an expression for all of the ℓ modes. By finding the coefficient of the terms that are linear in ℓ (the A term) and independent of ℓ (the B term), we can remove these terms from the expression and then analytically perform the sum from $\ell = 0$ to ∞. On the other hand, because we have the analytic expression for the $\tilde{S}$ field, we can actually pick out the pN expansion of the $D^{(2j)}_\alpha$ without making reference to the Hadamard expansions at all.[2]

6.3.2 Generating the $\tilde{S}$ Field for Large ℓ

The mode sum regularization involves a sum over m. In our case, we can use the (corrected)[3] relationship from Hikida et al. [4] given in their Eqs. (3.7-3.8)

$$\sum_{m=-\ell}^{\ell} m^{2j} |Y_{\ell,m}(\pi/2,\phi)|^2 = \lambda_j(\ell), \tag{6.3.2}$$

[2] This method is actually the analytical version of the numerical techniques used by Shah et al. [3][50]. In his work, Shah computes many ℓ modes of the retarded field (84 modes in [50]) numerically, and, knowing that the renormalized force falls off faster than any power of ℓ, plotted the results and determined successive regularization parameters by fitting for the ℓ dependence. It should also be noted that if the D term (finite remainder) was non-zero, then it would be necessary to use the local expansions to determine its value.

[3] This correction was found by Eric Van Oeveren

where the $\lambda_j(\ell)$ can be found by performing a Taylor series about $z=0$ of the following expression,

$$\sum_{n=0}^{\infty}\frac{\lambda_n(\ell)z^{2n}}{(2n)!}=\frac{2\ell+1}{4\pi}e^{\ell z}{}_2F_1\left(1/2,-\ell;1;1-e^{-2z}\right), \qquad (6.3.3)$$

and equating the coefficients for each order of z^2. We also note that the sum over m of $m^{2j+1}|Y_{\ell,m}(\pi/2,\phi)|^2$ will vanish.

Following Hikida [4], we return to the Green's function and rewrite it as

$$g^{\tilde{S}}_{\ell,m,\omega}(r_<,r_>)=\sum_{j=0}^{\infty}G_{\ell,m,k}(r_<,r_>). \qquad (6.3.4)$$

Because of this, we can perform the inverse Fourier transform using

$$\int d\omega\omega^{2k}e^{-i\omega(t-t')}=2\pi\left(i\partial_{t'}\right)^{2k}\delta(t-t'), \qquad (6.3.5)$$

which means we can write the force $F^{\tilde{S}}_{\alpha,\ell}$ *in the time domain* as

$$F^{\tilde{S}}_{\alpha,\ell}(t,r,\theta,\phi)=q^2\lim_{x\to z}\nabla_\alpha\sum_{m,k}(i\partial_t)^{2k}u^tG_{\ell,m,k}(r,z^r(t)Y_{\ell,m}(\theta,\phi)\bar{Y}_{\ell,m}(z^\theta(t),z^\phi(t)). \qquad (6.3.6)$$

Here we have replaced the notation used throughout most of this document with the argument of the Green's function being (r,r') instead of $(r_<,r_>)$. This is the only place where we will use the notation (r,r').

In the case of circular orbits, the ∂_t^{2k} can just be replaced by $(m\Omega)^{2k}$, but it is worth pausing here to consider the implications of Hikida's split. Using this method, it will be possible to renormalize analytically for an *arbitrary* trajectory in the time domain to a given pN order. By renormalizing this way, the rest of the field is C^∞, meaning we would only need to calculate a limited number of ℓ values to get the desired pN order.

First, we return to Eqs (5.4.5-5.4.10), and, using the definitions for the Wronskian, $W_{\ell m\omega}$, ϕ_c^ν, and $\phi_c^{-\nu-1}$, we write the Green's function $g^{\tilde{S}}_{\ell m\omega}(r_<,r_>) = -W^{-1}_{\ell m\omega}\phi_c^\nu(r_<)\phi_c^{-\nu-1}(r_>)$.

Second, we take the gradient of this expression and evaluate it at $r_<=r_>=r_0$, finding the values with the derivatives evaluated both above and below the particle (i.e. we compute the derivatives $\partial_{r_<}$ and $\partial_{r_>}$). Third, we make the substitution $\omega\to m\Omega$, and use Eq. (6.3.2) to sum over the m-modes.

This results in two quantities $F^{\tilde{S}}_{r,\ell>} := \partial_{r_>}\Phi^{\tilde{S}}$ and $F^{\tilde{S}}_{r,\ell<} := \partial_{r_<}\Phi^{\tilde{S}}$, which we use to define in the antisymmetric and symmetric parts of the $\tilde{S}$ force respectively,

$$\begin{aligned} F^{\tilde{S}}_{\alpha,\ell-} &= \frac{1}{2}\left(F^{\tilde{S}}_{\alpha,\ell>} - F^{\tilde{S}}_{\alpha,\ell<}\right) \\ F^{\tilde{S}}_{\alpha,\ell+} &= \frac{1}{2}\left(F^{\tilde{S}}_{\alpha,\ell>} + F^{\tilde{S}}_{\alpha,\ell<}\right). \end{aligned} \tag{6.3.7}$$

The antisymmetric piece is

$$\begin{aligned} F^{\tilde{S}}_{r,\ell-} &= -\frac{q^2 u^t(2\ell+1)}{8\pi r^2}\sum_{n=0}^{6}\left(\frac{2M}{r}\right)^n + O(M/r)^7 \\ &= -\frac{q^2(2\ell+1)}{8\pi u^t r^2}\frac{1}{1-\frac{2M}{r}}, \end{aligned} \tag{6.3.8}$$

which is precisely what we expect from Eq. (3.4.42) for the A_α term, up to the factor of 4π, which arises due to the different conventions used in the derivation used in Chapter 3, and those we adopted in chapters 5 and 6.

Turning to $F^{\tilde{S}}_{\alpha,\ell+}$, we know that this term must be dominated by a term that is independent of ℓ, i.e. the B_α term. Thus, we can determine the B_α term simply by taking the limit as $\ell \to \infty$, which leads to

$$\begin{aligned} \lim_{\ell\to\infty} F^{\tilde{S}}_{r,\ell+} = & -\frac{q^2}{8\pi r^2 u^t}\Bigg[1 + \left(\frac{-(\Omega r)^2}{4} + \frac{2M}{r}\right) + \left(\frac{-27(\Omega r)^4}{64} + \frac{9M^2}{2r^2}\right) \\ & + \left(\frac{-125}{256}(\Omega r)^6 - \frac{45M}{32r}(\Omega r)^4 + \frac{15M^2}{8r^2}(\Omega r)^2 + 10\frac{M^3}{r^3}\right) \\ & + \Bigg(\frac{-8575(\Omega r)^8}{16384} - \frac{175M}{64r}(\Omega r)^6 - \frac{315M^2}{128r^2}(\Omega r)^4 + \frac{35M^3}{4r^3}(\Omega r)^2 \\ & + \frac{175M^4}{8r^4}\Bigg) + \Bigg(-\frac{35721}{65536}(\Omega r)^{10} - \frac{33075M}{8192r}(\Omega r)^8 - \frac{4725M^2}{512r^2}(\Omega r)^6 \\ & + \frac{945M^4}{32r^4}(\Omega r)^2 + \frac{189M^5}{4r^5}\Bigg) + \Bigg(-\frac{586971(\Omega r)^{12}}{1048576} - \frac{43659M}{8192r}(\Omega r)^{10} \\ & - \frac{606375M^2}{32768r^2}(\Omega r)^8 - \frac{5775M^3}{256r^3}(\Omega r)^6 + \frac{10395M^4}{512r^4} + \frac{693M^5}{8r^5}(\Omega r)^2 \\ & + \frac{1617M^6}{16r^6} + pN7\Bigg]. \end{aligned} \tag{6.3.9}$$

This is the B_r term to 6 pN orders. Now that we have identified the A_r and B_r terms, (notice that we did so purely by analyzing the solution to the retarded field, only making reference to Eqs. (3.4.42) and (3.4.25) to check our solutions), we can write $F^{\tilde{S}-S}_{r,\ell}$ by subtracting Eqs. (6.3.8) and (6.3.9) from the expression for $F^{\tilde{S}}_{r,\ell}$. To ease writing, we will use the notation

$$\{m\} = [(2\ell+1+2m)(2\ell+1-2m)], \qquad (6.3.10)$$

which allows us to write,

$$\begin{aligned} F^{\tilde{S}-S}_{r,\ell} &= \frac{q^2}{8\pi r^2}\left[\frac{3(\Omega r)^2}{4\{1\}} + \left(\frac{9(\Omega r)^4}{64}\frac{184\ell(\ell+1)-135}{\{1\}\{2\}} - \frac{3M^2}{2r^2\{1\}}\right)\right. \\ &+\left(\frac{25(\Omega r)^6}{256}\frac{7875+4\ell(\ell+1)(-3247+900\ell(\ell+1))}{\{1\}\{2\}\{3\}}\right. \\ &+\frac{5M(\Omega r)^4}{32r}\frac{618+\ell(\ell+1)(-901+72\ell(\ell+1))}{(\ell-1)(\ell+2)\{1\}\{2\}} \\ &\left.\left.+\frac{(\Omega r)^2 M^2}{8r^2}\frac{135+4\ell(\ell+1)(-89+48\ell(\ell+1))}{(\{0\}\{1\}\{2\})^2} - \frac{6M^3}{r^3\{1\}}\right) + pN4.\right. \end{aligned} \qquad (6.3.11)$$

We compute this out to 6 pN orders, but due to the length of the expressions, we will not include them here, as three pN orders are enough to demonstrate the procedure. This also raises a few apparent paradoxes, which we will address in rising order of complexity.

When we subtract only the A_r and B_r terms from the mode-sum of the retarded field, we do not recover a C^∞ field, and so the resulting expression for the modes do not fall off faster than any power in ℓ. But this leads to the next objection: if we the last two lines of Eq. (6.3.11) then we see two terms (the $M\Omega^4$ and $M^2\Omega^2$ terms) whose denominators are not of the form of a vanishing sum. After all, our higher-order regularization parameters are supposed to be of the form $Const(\prod_{k=1}^{N}\{k\}^{-1}$.

We can rewrite anything of the form

$$\begin{aligned} a\ell(\ell+1)+b &= \frac{a}{4}\left(4\ell^2+4\ell+\frac{4b}{a}\right) \\ &= \frac{a}{4}\left((4\ell^2+4\ell+(1+2m)(1-2m))+\frac{4b}{a}-(1+2m)(1-2m)\right) \\ &= \frac{a}{4}\left((2\ell+1+2m)(2\ell+1-2m)+\frac{4b}{a}-(1+2m)(1-2m)\right) \\ &= \frac{a}{4}\left(\{m\}+\frac{4b}{a}-(1+2m)(1-2m)\right) \end{aligned} \qquad (6.3.12)$$

Thus

$$\begin{aligned} 184\ell(\ell+1)-135 &= 46\{2\}-135-46(-15) \\ &= 46\{2\}+555; \end{aligned} \qquad (6.3.13)$$

and

$$\begin{aligned}\frac{9(\Omega r)^4}{64}\frac{184\ell(\ell+1)-135}{\{1\}\{2\}} &= \frac{9(\Omega r)^4}{64}\frac{46\{2\}+555}{\{1\}\{2\}} \\ &= \frac{9(\Omega r)^4}{64}\left(\frac{46}{\{1\}}+\frac{555}{\{1\}\{2\}}\right). \qquad (6.3.14)\end{aligned}$$

Now, we can see that this is clearly the sum of a $D_r^{(2)}$ and $D_r^{(4)}$.

Now, using this technique every term in Eq. (6.3.11) can be rewritten in the form of a constant divided by an even polynomial in ℓ. By judiciously choosing which $[m]$ we use, we can write all of the terms that are part of vanishing sums so that they have the form of $D^{(2j)}(\prod_{k=1}^{j}\{k\})^{-1}$.

The only terms that cannot be written this way (at this order) are the 3pN terms of order M and M^2. We will focus on the second term first.

The M^2, 3pN term from $F_r^{\tilde{S}-S}$ will contain a term of the form $(\{0\}\{1\}\{2\})^{-2}$, which falls off as ℓ^{-6}, but does not sum to zero. [4] Since we know there is no D_α term, this *must* be part of the actual force. On the other hand, since this falls off as a finite power in ℓ, it cannot be part of the force.

This apparent contradiction is solved by realizing that this term must be a sum of a piece that falls off faster than any power and contributes to the force, and a piece that falls off as a finite power of ℓ that is an element of a vanishing sum. To demonstrate how this happens, consider the simpler case of a term that goes as $(2\ell+1)^2$.

$$\frac{1}{(2\ell+1)^2}-\frac{4^1(1/4)}{(2\ell+3)(2\ell-1)}=\frac{-4}{(2\ell+1)^2[(2\ell+3)(2\ell-1)]}=\frac{-(1/4)4^2}{\{0\}\{1\}}, \qquad (6.3.15)$$

and so the $D^{(2)}$ term from $(2\ell+1)^{-2}$ is equal to $1/4$. Now, clearly the $D^{(4)}$ term is -1/4. Subtracting this term off gives us

$$\frac{4^3(1)}{\{0\}\{1\}\{2\}},$$

so $D^{(6)}=1$. Continuing this procedure gives us $D^{(8)}=-9$, $D^{(10)}=144$, $D^{(12)}=-3600$,

[4] in fact, it sums to $3\pi^2/256$. These terms are one of the sources for the π^2 terms that appear in our final answer.

and $D^{(14)} = 129600$. After subtracting all of these terms we are left with

$$\frac{-6350400(4^8)}{\{0\}\{1\}\{2\}\{3\}\{4\}\{5\}\{6\}\{7\}}. \tag{6.3.16}$$

Therefore, for any finite power of ℓ, we can continue this procedure. Summing over the remainder still gives $\pi^2/8$, so we have not changed the result of the summation. Furthermore, if we add the $\ell = 0$ and $\ell = 1$ values of the expression in Eq. (6.3.16), we recover a relative error of 0.08. Summing the first seven values ($\ell = 0$ through $\ell = 6$), we recover a relative error of 3.7×10^{-6}.

We will use similar tricks with the order M term. Notice that $(\ell - n)(\ell + 1 + n) = \ell(\ell + 1) - n(n + 1)$, and

$$\begin{aligned} \frac{1}{(\ell-n)(\ell+1+n)} &- \frac{4}{(2\ell+1+2m)(2\ell+1-2m)} \\ &= \frac{1+4[n(n+1)-m^2]}{(\ell-n)(\ell+1+n)(2\ell+1+2m)(2\ell+1-2m)} \end{aligned} \tag{6.3.17}$$

Notice that, even though the $(\ell - 1)(\ell + 2)$ denominator of the order M term, blows up at $\ell = 1$ (which is acceptable for the retarded field since at 3pN, we expect this term to be valid only for $\ell \geq 4$), we can still use it to identify the regularization parameters.

Therefore, we can renormalize these terms as well, following a similar procedure to that of the M^2 term.

So, even though Eq. (6.3.11) may not appear to have the exact form we were hoping for, it can be written so that, for a given n_{max} the $\tilde{S}$ field can be split into terms that either match the form of the singular field, allowing us to identify the A_α, B_α and the $D^{(2)}$ through $D^{(n_{max})}$ terms which we recognize as the singular field, or into terms that fall of faster than $\ell^{n_{max}}$, which will have terms that contribute to the force.

We do not need to find these higher order regularization parameters, as we can perform the sum over all ℓ.

6.3.3 The Value of the $\tilde{S} - S$ Field

By using the explicit values for $\ell = 0$ through $\ell = 7$ and then using the general formula for $\ell = 8$ to infinity we can perform the full renormalization.

We find the $F_r^{\tilde{S}-S}$ force to be

$$\begin{aligned}\frac{F_r^{\tilde{S}-S}}{\frac{q^2}{4\pi r^2}} &= -\left[\frac{5(r^3\Omega^2)}{19M}+\frac{2}{7}\right]-\left[\frac{1417r^5\Omega^4}{3002M}-\frac{89(r\Omega)^2}{133}-\frac{4M}{7r}\right]\\ &-\left[\frac{9804331(r^7\Omega^6)}{14205464M}-\frac{136153(\Omega r)^4}{42028}+\frac{5(\Omega r)^2M}{38r}-\frac{9M^2}{7r^2}\right]\\ &-\left[\frac{1015083323057\Omega^8r^9}{1108452355920M}-\frac{1243509067(\Omega r)^6}{127849176}\right.\\ &\left.+\frac{866709919(\Omega r)^4M}{97534980r}+\frac{(\Omega r)^2M^2}{r^2}\left(\frac{3949}{5054}-\frac{7\pi^2}{64}\right)-\frac{482M^3}{133r^3}\right]\\ &+pN4. \end{aligned} \tag{6.3.18}$$

We can compare this to the $\tilde{R}$ force from eq. (6.2.2)

$$\begin{aligned}\frac{F_r^{\tilde{R}}}{\frac{q^2}{4\pi r^2}} &= \left[\frac{5r^3\Omega^2}{19M}+\frac{2}{7}\right]+\left[\frac{1417r^5\Omega^4}{3002M}-\frac{89(r\Omega)^2}{133}-\frac{4M}{7r}\right]\\ &+ \left[\frac{9804331r^7\Omega^6}{14205464M}-\frac{136153(r\Omega)^4}{42028}+\frac{5rM\Omega^2}{38}-\frac{9M^2}{7r^2}\right]\\ &+ \left[\frac{1015083323057r^9\Omega^8}{1108452355920M}-\frac{1243509067(r\Omega)^6}{127849176}+\frac{3949\Omega^2M^2}{5054}-\frac{482M^3}{133r^3}\right.\\ &- \left.(Mr^3\Omega^4)\left(\frac{4}{3}(\gamma+\ln[2r\Omega])-\frac{93892831}{10837220}\right)\right]+pN4.\end{aligned}$$

As we expect, the pN 0, 1, and 2 terms cancel each other exactly. Adding these together, and focusing on the 3pN term only, we find

$$F_r^R=\frac{q^2}{4\pi r^2}\left[\frac{7\pi^2}{64}\left(\frac{M}{r}\right)^2v^2+\left(-\frac{2}{9}-\frac{4}{3}(\gamma+\ln(2v))\right)\frac{M}{r}v^4\right]+pN4. \tag{6.3.19}$$

By just focusing on the first non-vanishing order, we can glean some useful information regarding the splitting of the fields. Notice that the M^0 and the v^0 terms from the $\tilde{S}$ and $\tilde{R}$ expressions cancel each other out exactly. This could be predicted, since we know that the conservative self-force vanishes both in flat spacetime and in Kerr spacetime for a static scalar charge.

Also notice how $f_r^{\tilde{S}-S}$ does actually contribute to the force, confirming that $f_r^{\tilde{S}} \neq f_r^S$. Another key feature to notice is that the $\ln[2v]$ and γ terms come straight from the $f_r^{\tilde{R}}$ expression (at higher orders there are also polygamma terms $\phi^(n)(x)$, which originate from the $F_r^{\tilde{R}}$ expression). This is to be expected– the $\Phi^{\tilde{R}}$ field contains all of the $\Gamma[r]$ functions, the derivatives of which are responsible for the appearance of the γs and $\psi^{(n)}$s.

Now we can write the full expression for the conservative self-force.

To sixth post Newtonian order, we find

$$
\begin{aligned}
F_r^R \quad = \quad & \frac{q^2}{4\pi r^2}\Bigg\{\left[\frac{7\pi^2}{64}\left(\frac{M}{r}\right)^2 v^2 + \left(-\frac{2}{9} - \frac{4}{3}(\gamma + \ln(2v))\right)\frac{M}{r}v^4\right] \\
& + \Bigg[\left(2 + \frac{7\pi^2}{64}\right)\left(\frac{M}{r}\right)^3 v^2 + \left(\frac{7}{9} - \frac{83\pi^2}{1024} + \frac{8}{3}(\gamma + \ln(2v))\right)\left(\frac{M}{r}\right)^2 v^4 \\
& + \left(\frac{479}{45} - \frac{128}{15}\ln(2) - \frac{22}{3}(\gamma + \ln(2v))\right)\frac{M}{r}v^6\Bigg] + \left[-\frac{38\pi}{45}\left(\frac{M}{r}\right)^2 |v|^5\right] \\
& + \Bigg[\left(4 + \frac{11\pi^2}{256}\right)\left(\frac{M}{r}\right)^4 v^2 + \left(-\frac{19}{9} - \frac{141\pi^2}{1024} - \frac{2}{3}(\gamma + \ln(2v))\right)\left(\frac{M}{r}\right)^3 v^4 \\
& + \left(-\frac{517}{15} + \frac{1529\pi^2}{2048} + \frac{512}{15}\ln(2) + 36(\gamma + \ln(2v))\right)\left(\frac{M}{r}\right)^2 v^6 + \Bigg(\frac{54647}{1260} \\
& - \frac{1216}{105}\ln(2) - \frac{2187}{70}\ln(3) - \frac{119}{6}(\gamma + \ln(2v))\Bigg)\frac{M}{r}v^8\Bigg] \\
& + \left[\frac{76\pi}{45}\left(\frac{M}{r}\right)^3 |v|^5 - \frac{1783\pi}{315}\left(\frac{M}{r}\right)^2 |v|^7\right] \\
& + \Bigg[\left(\frac{341}{45} - \frac{23\pi^2}{256} + \frac{16}{3}\ln\left(\frac{2M}{r}\right)\right)\left(\frac{M}{r}\right)^5 v^2 + \Bigg(-\frac{41}{18} + \frac{5221\pi^2}{2048} \\
& - \frac{76585\pi^4}{524288}\Bigg)\left(\frac{M}{r}\right)^4 v^4 + \Bigg(\frac{6239}{500} - \frac{138469\pi^2}{30720} - \frac{1984}{45}\ln(2) \\
& - \frac{49207}{675}(\gamma + \ln(2v)) + \frac{152}{45}(\gamma + \ln(2v))^2 + \frac{8}{3}\psi^{(2)}(2)\Bigg)\left(\frac{M}{r}\right)^3 v^6 \\
& + \Bigg(-\frac{803219}{2520} + \frac{423951\pi^2}{131072} + \frac{1408}{315}\ln(2) + \frac{6561}{35}\ln(3) \\
& + \frac{5131}{27}(\gamma + \ln(2v))\Bigg)\left(\frac{M}{r}\right)^2 v^8 + \Bigg(\frac{7319647}{68040} - \frac{35920}{189}\ln(2) \\
& + \frac{8019}{140}\ln(3) - \frac{161}{4}(\gamma + \ln(2v))\Bigg)\frac{M}{r}v^{10}\Bigg] + pN6.5\Bigg\},
\end{aligned}
\tag{6.4.1}
$$

In the geodesic limit, we find

$$F^R_{r,geo} = \frac{q^2v^6}{4\pi r^2}\Bigg\{\left(\frac{7\pi^2}{64}-\frac{2}{9}-\frac{4}{3}(\gamma+\ln[2v])\right)+\left(\frac{604}{45}+\frac{29\pi^2}{1024}-\frac{128}{15}\ln[2]\right.$$
$$\left.-\frac{14}{3}(\gamma+\ln[2v])\right)v^2-\frac{38\pi}{45}|v|^3+\left(\frac{1511}{140}+\frac{31}{2}(\gamma+\ln[2v])\right.$$
$$\left.+\frac{1335\pi^2}{2048}+\frac{320}{7}\ln[2]-\frac{2187\ln[3]}{70}\right)v^4-\frac{139\pi|v|^5}{35}+\left(\frac{-41117659}{212625}\right.$$
$$+\frac{152}{45}(\gamma+\ln[2v])^2+\frac{2332769\pi^2}{1966080}-\frac{76585\pi^4}{524288}+\frac{69199\gamma}{900}+\frac{78799}{900}\ln[2v]$$
$$\left.+\ \frac{34263}{140}\ln[3]-\frac{95174}{405}\ln[2]+\frac{8}{3}\psi^{(2)}[2]\right)v^6+O(v^7)\Bigg\} \qquad (6.4.2)$$

While Eq. (6.4.1) is far from pleasing to the eye, we can understand the value of this equation when we look at the far simpler Eq. (6.4.2). Even keeping only terms out to v^{12}, we can see that the expressions are beginning to become very unwieldy. In order to produce an expression of sufficient accuracy to evolve an orbit, it will be necessary to keep many more terms, terms which will grow more and more complicated as we increase our accuracy requirements.

In these circumstances, it is useful to be able to take limits of the expression to recover simpler, known results. Using Eq. (6.4.2), we can only take the limit as $v \to 0$. When we do so, we would also set M to zero, so that while we can take a limit, it simplifies to considering the self-force on a static charge in flat spacetime.

Starting from Eq. (6.4.1), it is possible to take the limit as $v \to 0$ while holding M fixed, or the limit as $M \to 0$ while holding v fixed. This allows us to check that the result for circular motion agrees with the results for a static particle in Schwarzschild, and for a particle moving on a circular orbit in flat space-time. [5]

The first two terms from Eq. (6.4.2) agree with Hikida et al. [5], but the rest of the terms are new[6], and so to compare them we will consider a few important limits from numerical studies.

[5]The fact that these limits give the same value as the self-force on a static charge in flat spacetime, namely $F^R_\alpha = 0$, means that we can still gain confidence in our result because it also agrees with two scenarios that are both more complicated than the static charge in flat spacetime.

[6]Hikida et al. needed to compute these terms numerically, but they did not write down their explicit forms.

6.4.1 Comparisons with Literature

We will compare with two numerical studies, namely that of Detweiler, Messerataki and Whiting [48], who analyzed circular orbits at $r = 10M$, and with that of Warburton et al. [45], who analyzed the self-force for several different values of the radius. We choose to compare with $r = 50M$, where one would imagine that v^6 correction terms will be sufficient to recover several digits of accuracy.

Table 1: Converging to DMW ($r = 10M$)

Power of v	F_r^R	Relative Difference
v^6	6.98505×10^{-6}	-0.4932
v^8	1.42163×10^{-5}	0.0313
v^9	1.33773×10^{-5}	-0.0295
v^{10}	1.50205×10^{-5}	0.0897
v^{11}	1.46263×10^{-5}	0.0611
v^{12}	1.37594×10^{-5}	-0.0018
DMW	$1.378448171 \times 10^{-5}$	—

Table 1: We demonstrate how we approach the results from DMW for $r = 10M$, $q^2 = 4\pi$, $M = 1$. It is interesting to note how the results from $O(v^9)$ are more accurate than either the $O(v^{10})$ and $O(v^{11})$ expressions.

For $r = 50M$, we can compare with the $a = 0$ values from Table III of Warburton et al. [45].

The agreement we find with these two studies is promising in the sense that we are converging to the expected values and the convergence is at the anticipated rates (very quickly at $r = 50M$ and more slowly for $r = 10M$). The work by DMW required the computation of 41 explicit ℓ-modes, followed by the use of an approximation method for higher ℓ to speed convergence. Warburton et al., used 56 ℓ-modes.

In the method we have used, it is perhaps correct to say that we computed either 9 ℓ-modes, or all of them, in the sense that we compute 8 modes explicitly ($\ell = 0$

Table 2: Converging to Warburton ($R = 50M$)

Power of v	F_r^R	Relative Difference
v^6	5.66868×10^{-9}	-0.10683
v^8	6.37183×10^{-9}	3.95878×10^{-3}
v^9	6.34781×10^{-9}	1.75506×10^{-4}
v^{10}	6.35288×10^{-9}	9.7452×10^{-4}
v^{11}	6.35063×10^{-9}	6.18665×10^{-4}
v^{12}	6.34664×10^{-9}	-8.9682×10^{-6}
Warburton	6.3467×10^{-9}	—

Table 2: We demonstrate how we approach the results from Warburton et al. for $r = 50M$, $q^2 = 4\pi$, $M = 1$. It is interesting to note how once again the results from $O(v^9)$ are more accurate than either the $O(v^{10})$ and $O(v^{11})$ expressions. Also note that the relative difference for v^{12} is meaningless, since Warburton only included 5 significant figures.

through $\ell = 7$) and then we compute the expression for general ℓ. While it is unarguably more computationally expensive to compute a general ℓ term then it is to compute an individual term numerically, we only need to compute one term for all ℓ above our desired pN accuracy and we can perform the sum over m analytically and do not need to compute each m mode separately.

Since we regularize analytically, we do not need to do any fitting of the $D_\alpha^{(2j)}$ terms required in numerical analysis. If the analytic summation of the $D_\alpha^{(2j)}$ terms becomes awkward, we can pick these terms out by eye– After subtracting the A_α and B_α terms from $F_\alpha^{\tilde{S}}$, we can multiply the resulting term by $4\ell^2$, and take the limit as $\ell \to \infty$. This result will be the $D_\alpha^{(2)}$ term. This process can be repeated ad infinitum until we find a term that sums easily.

6.5 Conclusions

In this dissertation, we have explored the details of self-force regularization for accelerated particles. Despite the fact that there are unlikely to be many systems of astrophysical interest that experience significant acceleration (at zeroth order in the mass ratio), considering accelerated motion can lead to significant insights. A single glance at an expression like that in Eq. (6.4.2) should be enough to convince the reader that as we pursue higher pN orders where the expressions will become even messier, it will be useful to have a number of limiting scenarios we can check explicitly.

Furthermore, the expressions for an accelerated charge in the frequency domain are identical to those for elliptic orbits so that as we advance to study these more complicated orbits, we can be confident in our frequency domain terms as we have a ready-made check at each point of the calculation, wherein we can compare our equations to the corresponding ones in three different limiting cases as a sort of sanity check.

In addition, we have helped pave the way for answering whether the self-force might act as a cosmic censor, providing the necessary tools to renormalize the self-force for massive, charged particles moving in electrovac. This same work has laid the foundations for extending the renormalization of the gravitational self-force beyond vacuum spacetimes.

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RESEARCH INTERESTS

Gravitational wave physics, gravitational self-force, black hole perturbation theory, binary black hole systems, extreme mass-ratio inspirals (EMRIs)

TEACHING EXPERIENCE

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ORGANIZATIONS

2013–present Member of the American Physical Society
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SELECTED CONFERENCES AND SEMINARS

"Perturbing Black Holes, Dividing by Zero, and Getting away with it too." Truman State University, Kirksville, Missouri, USA. 20 May 2015. (contributed talk)

"Self-Force on Accelerated Particles in Schwarzschild." American Physical Society April Meeting, Baltimore, Maryland, USA. 14 April 2015. (contributed talk)

"Combined Gravitational and Electromagnetic Self-Force in Electrovac." Midwest Relativity Meeting, Oakland University, Rochester, Michigan. 8 November 2014. (contributed talk)

"Combined Gravitational and Electromagnetic Self-Force in Electrovac." CAPRA 17, California Institute of Technology, Pasadena, California. 25 June 2014. (contributed talk)

"Self-Force on Accelerated Particles in Generic Spacetimes" American Physical Society April Meeting, Denver, Colorado, 16 April 2013. (contributed talk)

"Self-Force on Accelerated Particles in Generic Spacetimes" CAPRA 15, University of Maryland College Park, College Park, Maryland, 11 June 2012. (contributed talk)

"Mother Guth: Investigating the Progeny of false-vacuum bubbles" UW Milwaukee Center for Gravitation and Cosmology Seminar, Milwaukee, Wisconsin. 2 October 2009. (contributed talk)

PUBLICATIONS

T.M. Linz , E. VanOeveren, A.G. Wiseman, "Scalar Self-Force on Accelerated Particles in Schwarzschild", in preparation.

T.M. Linz , J. Friedman, A.G. Wiseman, "Combined gravitational and electromagnetic self-force on charged particles in electrovac spacetimes", Physical Review D **90**, 084031 (2014).

T.M. Linz , J. Friedman, A.G. Wiseman, "Self-force on an accelerated particle", Physical Review D **90**, 024064 (2014).